Seminar Lecture Note : Convex Optimization

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Contents

1 Introduction

The main goal of the convex optimization is to solve a minimization problem :

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad f(x)$$

where the function f is a convex function. Sometimes, we can set some constraints on the problem :

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & x \in C \end{aligned}$$

where C is a nonempty closed convex set.

1.1 Gradient Descent Method

Before we start, let's look into a gradient descent method, a famous method for minimization problem, to get an insight about steps of convex optimization.

The gradient descent method solves the problem

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad f(x)$$

where the function f is a differentiable convex function. Such problem is equivalent to

$$\underset{x}{\text{find}} \quad 0 = \nabla f(x) \quad \Leftrightarrow \underset{x}{\text{find}} \quad x = (\mathbb{I} - \alpha\nabla f(x))x \quad .$$

Thus, the minimization problem is transformed to a problem of finding *fixed point* of an operator $\mathbb{I} - \alpha\nabla f(x)$. The gradient descent method takes form of

$$x^{k+1} = (\mathbb{I} - \alpha\nabla f)x^k.$$

With a suitable stepsize α, an operator $\mathbb{I} - \alpha\nabla f$ becomes a L-Lipschitz mapping with $L < 1$. With the help of the Fixed Point Theorem, the iteration converges as $x^k \to x^*$, where x^* is one of the fixed points of an operator $\mathbb{I} - \alpha\nabla f$. Thus, $f(x^k)$ converges to the minimum value of f.

2 Convex Functions

2.1 Convexity

Definition 2.1 (Convex function)**.** A function $f(x)$ is called *convex* if it satisfies following three conditions.

- $\operatorname{dom} f$, the domain of f, is a convex set.
- f is a function to $\mathbb{R}$, i.e. $\operatorname{range} f \subset \mathbb{R}$.
- f satisfies the Jensen's Inequality :

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y), \quad \forall \theta \in [0,1].$$

Additionally, a function f is called *concave* if $-f$ is convex. To keep the notations simple, we will consider the value of $f(x)$ when $x \notin \operatorname{dom} f$ as ∞, where $a < \infty$ for all $a \in \mathbb{R}$. The Jensen's Inequality still holds with such extra assumption.

Here's a list of basic convex function examples.

- An affine function $f(x) = a^T x + b : \mathbb{R}^n \to \mathbb{R}$.
- Exponential function $f(x) = e^{ax} : \mathbb{R} \to \mathbb{R}$.
- Powers of absolute $f(x) = |x|^p : \mathbb{R} \to \mathbb{R}$, when $p \geq 1$.
- Powers $f(x) = x^p : \mathbb{R}^{++} \to \mathbb{R}$, when $p \geq 1$ or $p \leq 0$.
- Negative logarithm $f(x) = \log x : \mathbb{R}^{++} \to \mathbb{R}$.
- Negative entropy $f(x) = x \log x : \mathbb{R}^{++} \to \mathbb{R}$.
- Norms $f(x) = \|x\|_p : \mathbb{R}^n \to \mathbb{R}$, with $p \in [1, \infty]$.
- Quadratic function $f(x) = x^T P x : \mathbb{R}^n \to \mathbb{R}$, when $P \succeq 0$.

One can also prove that the given function is convex using following theorems.

Theorem 2.2. *For a given function $f : \mathbb{R}^n \to \mathbb{R}$, f is convex if and only if $g_{x,v}$ is convex for all $x, v \in \mathbb{R}^n$. The function $g_{x,v} : \mathbb{R} \to \mathbb{R}$ is defined as*

$$g_{x,v}(t) = f(x + tv).$$

This theorem proves that the following example is convex.

Remark 2.3. A function on square matrix X defined as

$$\log \det(x)$$

is concave function.

Theorem 2.4 (First order condition)**.** *When a given function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable on* $\operatorname{dom} f$*, f is convex if and only if*

$$f(y) \geq f(x) + \nabla f(x)^T (y - x), \quad \forall x, y \in \mathbb{R}^n.$$

Theorem 2.5 (Second order condition)**.** *When a given function $f : \mathbb{R}^n \to \mathbb{R}$ is twice differentiable on* $\operatorname{dom} f$*, f is convex if and only if*

$$\nabla^2 f(x) \succeq 0, \quad \forall x \in \mathbb{R}^n.$$

2.2 Convexity preserving Operations

In this subsection, we review some operations that preserves convexity.

Theorem 2.6 (Convexity preserving Operations)**.** *When the functions $f(x)$, $f_i(x)$ and $f(x,y)$ are convex functions of x for all $i = 1, 2, \cdots$ and y, then the all of the following functions are also convex.*

- *Nonnegative weighted sum :*

$$\sum_i w_i f_i(x), \quad \int w(y) f(x,y) dy,$$

  when $w_i, w(y) \geq 0$ for all $i = 1, 2, \cdots$ and y.
- *Pointwise Maximum :*

$$\max_i f_1(x), f_2(x), \cdots, f_m(x), \quad \sup_y f(x,y).$$

- *Composition to Affine function :*

$$f(Ax + b)$$

- *Minimization : Further assume that $f(x,y)$ is convex in (x,y). Then,*

$$\inf_{y \in C} f(x,y)$$

  is convex for convex set C.
- *Perspective function : Define a function $g(t,x)$ as*

$$g(t,x) = tf(x/t), \quad \operatorname{dom} g = \{(t,x) : x/t \in \operatorname{dom} f, t > 0\}.$$

  Then, $g(t,x)$ is convex if and if only when $f(x)$ is convex.

When the function is defined as a composition of convex and concave functions, here's the result on its convexity.

Theorem 2.7. *Define a function f of the form*

$$f(x) = h(g_1(x), g_2(x), \cdots, g_m(x))$$

with h nondecreasing on each elements, then

- *f is convex if all g_i and h are convex.*
- *f is concave if all g_i and h are concave.*

Using convexity preserving operation, we can prove that the following functions are convex.

- Log sum exponential : $f(x) = \log \sum_i e^{x_i} : \mathbb{R}^n \to \mathbb{R}$.
- Quadratic over linear : $f(x,y) = x^2/y : \mathbb{R}^2 \to \mathbb{R}$.

- Spectral : $f(X) = \left[\lambda_{\max}(X^T X)\right]^{1/2} : \mathbb{R}^{m\times n} \to \mathbb{R}$.

But, the most important convex function is the *conjugate* function of f. When the function f, which doesn't have to be a convex function, its conjugate function is defined as following.

Corollary 2.8 (Conjugate function)**.** *For a function f, its conjugate function f^* is defined as :*

$$f^*(y) = \sup_x \left\{x^T y - f(x)\right\}.$$

Regardless of the convexity of f, $f^*(y)$ is always convex. Furthermore, we have the inequality (called Fenchel's Inequality) :

$$f(x) + f^*(y) \geq x^T y.$$

2.3 CCP Functions

In this subsection, we handle more specific class of functions, called *CCP functions*. CCP stands for Convex, Closed, Proper function. Before defining CCP function, let's define an epigraph of function.

Definition 2.9 (Epigraph)**.** An epigraph of a given function $f(x)$ is defined as

$$\operatorname{epi} f = \{(x,t) : x \in \operatorname{dom} f, f(x) \leq t\}.$$

An epigraph is one way of expressing a single-valued function as a set. Note that a convexity of a function coincides with a convexity of the epigraph.

Theorem 2.10. *A single-valued function f is convex if and if only when the epigraph of f is a convex set.*

The theorem above is a direct consequence of the Jensen's Inequality.

Now we define a *closed* function and *proper* function.

Definition 2.11. A function f is called *closed* when its epigraph $\operatorname{epi} f$ is a closed set. A function f is called *closed* when the $f(x) > -\infty$ for all x and there exists some x that $f(x) < \infty$. In other words, a function f is *proper* if $\operatorname{epi} f$ is nonempty set without any vertical lines.

Since we have all the definition of *convex, closed, proper*, let's define a new class of functions, *CCP*.

Definition 2.12 (CCP functions)**.** A single-valued function f is called *CCP*, if it is convex, closed, and proper function. Equivalently, a function f is CCP if and if only when $\operatorname{epi} f$ is convex, closed, nonempty set without any vertical lines included.

Theorem 2.13. *When a function f is CCP, then the conjugate of the conjugate f is identical to f.*

$$f = f^{**}$$

Proof. Define a pointwise supremum of affine underestimators :

$$\tilde{f}(x) = \sup\{g(x) : g \text{ is affine}, g(z) \leq f(z) \text{ for all z}\}.$$

From the closedness, we have $f = \tilde{f}$, and $f^{**} = \tilde{f}$ from the definition. □

An indicator function is one useful example of CCP function.

Definition 2.14 (Indicator function)**.** The indicator function δ_C of a nonempty closed convex set C is defined as

$$\delta_C(x) = \begin{cases} 0 & x \in C \\ \infty & x \notin C. \end{cases}$$

With the help of the indicator function, the convex optimization problem with constraints can be written as a simple convex optimization problem without constraints

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad f(x) + \delta_C(x) \quad .$$

Before moving on, take in mind that minimization problem can also written using argmin as

$$\operatorname{argmin} f,$$

if the focus is on the minimizing solution. The arguement minimum of the function is defined as

$$\operatorname{argmin} f = \{x : f(x) = \inf f\}$$

When the target function f is CCP, it is known that $\operatorname{argmin} f$ is a closed, convex set.

2.4 Strong convexity and Smoothness

Definition 2.15 (Strong convexity and Smoothness)**.** A CCP function f is called

- *μ-strongly convex* if $f(x) - \frac{\mu}{2}\|x\|^2$ is convex.
- *L-smooth* if $f(x) - \frac{L}{2}\|x\|^2$ is concave.

Note that when f is twice continuously differentiable, μ-strongly convex is equivalent to $\nabla^2 f \succeq \mu I$ and L-smooth is equivalent to $\nabla^2 f \preceq LI$.

2.5 Convex duality

So, how is conjugate of the function useful in convex optimization? In this subsection we will cover about dual problem using the conjugate function. First, let's start with convex-concave function and saddle point problem.

Definition 2.16 (Convex-concave function)**.** A function $L(x, u) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \cup \{\pm\infty\}$ is called *convex-concave function* if it satisfies :

- $L(\cdot, u)$ is a convex function of x, for each u,
- $L(x, \cdot)$ is a concave function of u, for each x.

The point (x^*, u^*) is called a *saddle point* if it satisfies :

$$L(x^*, u) \leq L(x^*, u^*) \leq L(x, u^*), \quad \forall x \in \mathbb{R}^n, \forall u \in \mathbb{R}^m.$$

A convex-concave function induces primal-dual problem. In the end, we aim to make our original convex problem as a primal problem, and let dual problem to help finding the solution.

Definition 2.17 (Primal-Dual problem and Dualities)**.** When a convex-concave function $L(x,u) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \cup \{\pm\infty\}$ is given, the *primal problem* is

$$\underset{x}{\text{minimize}} \quad \sup_u L(x,u) \quad , \quad p^* = \inf_x \sup_u L(x,u),$$

and the *dual problem* is

$$\underset{u}{\text{maximize}} \quad \inf_x L(x,u) \quad , \quad d^* = \sup_u \inf_x L(x,u).$$

Note that the *weak duality* ($d^* \leq p^*$) always holds, while the *strong duality* ($d^* = p^*$) does not. We say *total duality* holds when the primal and dual solutions exist with the strong duality. The total duality holds if and if only when the saddle point of the convex-concave function exists.

Example 2.18 (Fenchel-Rockafeller Dual)**.** When a convex function f, g and a linear map A is given, consider a *Lagrange function* $L(x,u)$:

$$L(x,u) = f(x) + \langle u, Ax \rangle - g^*(u).$$

The primal problem is

$$\underset{x}{\text{minimize}} \quad f(x) + g(Ax) \quad ,$$

and the dual problem is

$$\underset{u}{\text{maximize}} \quad -f^*(-A^T u) - g^*(u) \quad .$$

Furthermore, if $A\text{dom}\, f \cap \text{int dom}\, g \neq \emptyset$, then the strong duality holds.

Remark 2.19. As mentioned in the previous section, an optimization problem with constraints can be rewritten as a optimization problem of the sum of two functions :

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad f(x) + \delta_C(x) \quad .$$

Thus, it is quite straightforward that convex optimization problem with constraints takes a form of primal problem of some convex-concave function. Usually, such convex-concave function is called *Lagrange function*.

Now the question is the reason why such dual problem is important. The dual problem is helpful on development of the optimizer solver. The idea is to solve optimization problem on both primal and dual variables. While the details may vary on the specific solvers, the key idea is :

$$\begin{aligned} x^{k+1} &= \underset{x}{\text{argmin}}\, L(x, u^k) \\ u^{k+1} &= \underset{u}{\text{argmin}} -L(x^{k+1}, u). \end{aligned}$$

Note that for the fixed u_k, a minimization problem of $L(x, u_k)$ does not have any constraints, which is much easier to handle. On the other hand, the effect of constraints is handled in the dual variable, as a form of projection functions. By splitting target function optimization and constraints of the problem, we aim to find a saddle point, which will lead to the solution of the original optimization problem. The questions we will handle in the future sections will be

1. Methods - For each types of problem, how is the iterative algorithm formulated?
2. Convergence conditions - Does the method converges to the solution? In which condition is the algorithm guaranteed to converge?.

In the next two sections, we will cover the basic knowledge on each questions. We will first handle about operators and subgradient to gather the tools to develop various methods. Then, we will cover about fixed point iteration, a key to guaranteeing the convergence.

3 Operators

3.1 Set-Valued Operators

A *set-valued operator* is a generalized form of function. It allows the image of a value to be a set, instead of single valued outcome.

Definition 3.1 (Set-Valued Operators)**.** A *set-valued operator* from $\mathbb{R}^n$ to $\mathbb{R}^n$ is written as

$$\mathbb{T} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n, \quad \mathbb{T}x \subset \mathbb{R}^n.$$

For the remark, when $\mathbb{T}x$ for each x is always singleton or empty, it is what we called a function.

Now let's define some sets related to the operator.

- Domain : $\operatorname{dom} \mathbb{T} = \{x : \mathbb{T}x \neq \emptyset\}$.
- Image of C : $\cup_{x \in C} \mathbb{T}x$.
- Range : $\operatorname{range} \mathbb{T} = \cup_{x \in \mathbb{R}^n} \mathbb{T}x$.
- Graph : $\operatorname{Gra} \mathbb{T} = \{(x, u) : u \in \mathbb{T}x\} \subset \mathbb{R}^n \times \mathbb{R}^n$. We often say $\mathbb{T}$ as $\operatorname{Gra} \mathbb{T}$.
- Zero set : $\operatorname{Zer} \mathbb{T} = \{x : 0 \in \mathbb{T}x\}$

Furthermore, let's define other useful operators.

- Composition : $\mathbb{T}\mathbb{S}x = \mathbb{T}(\mathbb{S}x), \quad \mathbb{T}\mathbb{S} = \{(x, z) : \exists y, s.t.(x, y) \in \mathbb{S}, (y, z) \in \mathbb{T}\}$.
- Operator sum : $(\mathbb{T} + \mathbb{S})x = \{t + s : t \in \mathbb{T}x, s \in \mathbb{S}x\}$.
- Inverse : $\mathbb{T}^{-1} = \{(y, x) : (x, y) \in \mathbb{T}\}$. Note that $\operatorname{Zer} \mathbb{T} = \mathbb{T}^{-1}(0)$.
- Identity Operator : $\mathbb{I} = \{(x, x) : x \in \mathbb{R}^n\}$. Note that $\mathbb{T}^{-1}\mathbb{T} = \mathbb{I}$ if $\mathbb{T}^{-1}$ is single valued.
- Zero Operator : $\mathbf{0} = \{(x, 0) : x \in \mathbb{R}^n\}$.

3.2 Subgradient

From calculus, we have already defined a gradient of a differentiable function. Here, we aim to generalize gradient to apply on the convex function regardless of differentiability. Recall that from the first order condition,

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle, \quad \forall x, y \in \mathbb{R}^n.$$

Furthermore, for each x, $\nabla f(x)$ is the only g that satisfies

$$f(y) \geq f(x) + \langle g, y - x \rangle, \quad \forall y \in \mathbb{R}^n.$$

Now, let's generalize the gradient.

Definition 3.2 (Subgradient)**.** When f is a convex function, its *subgradient* at x is

$$\partial f(x) = \{g : f(y) \geq f(x) + \langle g, y - x \rangle, \forall y \in \mathbb{R}^n\}.$$

Note that $\partial f(x)$ is a closed convex set. A function f is differentiable at x if and if only when $\partial f(x)$ is a singleton. We call a function f is *subdifferentiable* at x if

$$\partial f(x) \neq \emptyset.$$

When the function f is CCP, then

$$\begin{cases} \partial f(x) = \emptyset & x \notin \operatorname{dom} f \\ \partial f(x) \neq \emptyset & x \in \operatorname{ri}(\operatorname{dom} f). \end{cases}$$

The notation ri refers to a relative interior of the set, which is a interior point of a given set when the affine hull of the set is chosen as the underlying space.

The notion of subgradient is important since the optimization problem can be transformed to a problem of finding zero.

$$x^* \in \operatorname*{argmin}_x f(x) \Leftrightarrow 0 \in \partial f(x^*)$$

Theorem 3.3 (Subgradient Identities)**.** *When f, g are CCP functions,*

- $\partial \alpha f = \alpha \partial f$ *if* $\alpha > 0$.
- $g(x) = f(Ax) \Rightarrow \partial g(x) \supseteq A^T \partial f(Ax)$. *Equality holds if* $\operatorname{range} A \cap \operatorname{ri}(\operatorname{dom} f) \neq \emptyset$.
- $\partial(f+g) \supseteq \partial f + \partial g$. *Equality holds if* $\operatorname{dom} f \cap \operatorname{int} \operatorname{dom} g \neq \emptyset$.
- *Fenchel's identity :* $(\partial f)^{-1} = \partial f^*$.

Proof of Fenchel's identity.

$$\begin{aligned} u \in \partial f(x) &\Leftrightarrow 0 \in \partial f(x) - u \\ &\Leftrightarrow x \in \operatorname*{argmin}_z f(z) - u^T z \\ &\Leftrightarrow -f(x) + u^T x = f^*(u) \\ &\Leftrightarrow -f^{**}(x) + x^T u = f^*(u) \\ &\Leftrightarrow x \in \partial f^*(u) \end{aligned}$$

□

One of the consequence of the Fenchel's identity is a relation between strong convexity and smoothness.

Corollary 3.4. *When f CCP, f is μ-strongly convex if and if only f^* is $(1/\mu)$-smooth.*

3.3 Monotone Operators

A monotone operator extends the notion of monotone function to a Hilbert space. The monotonicity of an operator is heavily related to a convexity. Such relation allows the optimization problem to be rewritten as an inclusion problem of monotone operator. But first, let's go through definitions.

Definition 3.5 (Monotone operator)**.** An operator $\mathbb{T}$ is called *monotone* if it satisfies

$$\langle u - v, x - y \rangle \geq 0, \quad \forall (x, u), (y, v) \in \mathbb{T}.$$

Furthermore, a monotone operator $\mathbb{T}$ is called *maximal* or *maximally monotone* if there are no such monotone operator $\mathbb{S}$ such that satisfies $\mathbb{T} \subsetneq \mathbb{S}$.

Theorem 3.6. *When f is a convex, proper function, then ∂f is a monotone operator. Furthermore, if f is CCP, then ∂f is a maximally monotone operator.*

Now let's define strongly monotone operators. The definition only modifies 0 of the right hand side.

Definition 3.7 (Strongly monotone operator)**.** An operator $\mathbb{T}$ is

- *μ-strongly monotone* or *μ-coersive* with $\mu > 0$ if

$$\langle u - v, x - y \rangle \geq \mu \|x - y\|^2, \quad \forall (x, u), (y, v) \in \mathbb{T}.$$

- *β-inverse strongly monotone* or *β-cocoersive* with $\beta > 0$ if

$$\langle u - v, x - y \rangle \geq \beta \|u - v\|^2, \quad \forall (x, u), (y, v) \in \mathbb{T}.$$

Note that $\mathbb{T}$ is β-cocoersive if and if only when $\mathbb{T}^{-1}$ is β-strongly monotone. Also, if $\mathbb{T}$ is β-cocoersive, then $\mathbb{T}$ is $1/\beta$-Lipschitz, thus single valued.
Furthermore, an operator $\mathbb{T}$ is called *maximal μ-strongly monotone* if there are no such μ-strongly monotone operator $\mathbb{S}$ such that satisfies $\mathbb{T} \subsetneq \mathbb{S}$. *Maximal β-cocoersive* is also defined similarly. $\mathbb{T}$ is maximal β-cocoersive if and if only when $\mathbb{T}^{-1}$ is maximal β-strongly monotone.

Theorem 3.8. *When a function f is CCP,*

- *f is μ-strongly convex if and if only when ∂f is μ-strongly monotone.*
- *f if L-smooth if and if only when ∂f is $1/L$-cocoersive.*

We now define a *monotone inclusion problem* for a monotone operator $\mathbb{A}$:

$$\underset{x}{\text{find}} \quad 0 \in \mathbb{A}x \quad .$$

When $\mathbb{A} = \partial f$, it solves the convex optimization problem.

3.4 Non-expansive operators

Definition 3.9 (L-Lipschitz)**.** An operator $\mathbb{T}$ is *L-Lipschitz* if

$$\|u - v\| \leq L \|x - y\|, \quad \forall (x, u), (y, v) \in \mathbb{T}.$$

Definition 3.10 (Non-expansive and θ-averaged)**.** An operator $\mathbb{T}$ is *non-expansive* if it is 1-Lipschitz. Note that Lipschitz operator is always single valued. Thus, $\mathbb{T}$ is *non-expansive* if

$$\|\mathbb{T}x - \mathbb{T}y\| \leq \|x - y\|, \quad \forall x, y \in \mathbb{R}^n.$$

An operator $\mathbb{T}$ is *contraction* if it is L-Lipschitz with $L < 1$:

$$\exists L < 1 \quad s.t. \quad \|\mathbb{T}x - \mathbb{T}y\| \leq L \|x - y\|, \quad \forall x, y \in \mathbb{R}^n.$$

An operator $\mathbb{T}$ is *θ-averaged* for $\theta \in (0, 1)$ if there exists some non-expansive $\mathbb{C}$ such that :

$$\mathbb{T} = (1 - \theta)\mathbb{I} + \theta \mathbb{C}.$$

3.5 Resolvents

In this subsection, we aim to build an averaged operator based on the given monotone operator.

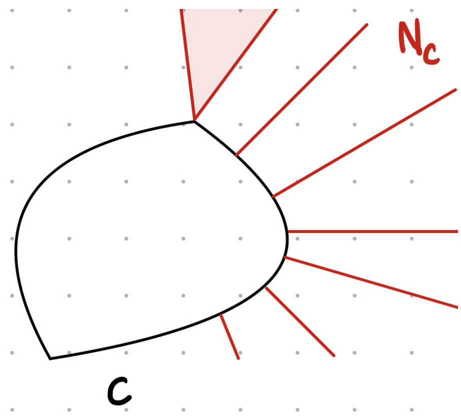
Definition 3.11 (Resolvent operator)**.** A *resolvent operator* of an operator $\mathbb{A}$ is

$$\mathbb{J}_{\mathbb{A}} = (\mathbb{I} + \mathbb{A})^{-1}.$$

A *reflection operator* of an operator $\mathbb{A}$ is

$$\mathbb{R}_{\mathbb{A}} = 2\mathbb{J}_{\mathbb{A}} - \mathbb{I}.$$

Example 3.12 (Projection and indicator function)**.** A subgradient of the indicator function is called *Normal cone operator* :



$$\partial \delta_C(x) = \mathbb{N}_C(x) = \begin{cases} \emptyset & x \notin C \\ \{y : \langle y, z - x \rangle \leq 0, \forall z \in C\} & x \in C. \end{cases}$$

Note that $\mathbb{N}_C = \alpha \mathbb{N}_C$ for any $\alpha > 0$. The resolvent of the normal cone operator is

$$\mathbb{J}_{\alpha \partial \delta_C} = \mathbb{J}_{\mathbb{N}_C} = \Pi_C,$$

where Π_C is a projection to a closed, convex set C.

Theorem 3.13 (Averagedness of Resolvent)**.** *When $\mathbb{A}$ is maximal monotone, $\mathbb{R}_{\mathbb{A}}$ is nonexpansive with* $\operatorname{dom} \mathbb{R}_{\mathbb{A}} = \mathbb{R}^n$*, and $\mathbb{J}_{\mathbb{A}}$ is $(1/2)$-averaged with* $\operatorname{dom} \mathbb{J}_{\mathbb{A}} = \mathbb{R}^n$*.*

Proof. For the nonexpansiveness, assume $(x, u), (y, v) \in \mathbb{J}_{\mathbb{A}}$. Then,

$$x \in u + \mathbb{A}u, \quad y \in v + \mathbb{A}v.$$

By monotonicity,

$$\langle (x - u) - (y - v), u - v \rangle \geq 0.$$

Hence,

$$\|(2u - x) - (2v - y)\|^2 \leq \|x - y\|^2.$$

Thus, $\mathbb{R}_{\mathbb{A}}$ is nonexpansive and $\mathbb{J}_{\mathbb{A}}$ is (1/2)-averaged. For the domain to be $\mathbb{R}^n$, it comes from *Minty's surjectivity theorem.* □

Note that domain being $\mathbb{R}^n$ is important since we will use the resolvent as an operator in fixed point iteration.

Remark 3.14. The zero set of a monotone operator $\operatorname{Zer}\mathbb{A}$ is identical to the fixed point set of the resolvent $\operatorname{Fix}\mathbb{J}_{\mathbb{A}}$.

$$0 \in \mathbb{A}x \Leftrightarrow x \in x + \mathbb{A}x \Leftrightarrow \mathbb{J}_{\mathbb{A}}x = x.$$

Also, $\operatorname{Zer}\mathbb{A}$ is closed, convex set if $\mathbb{A}$ is maximal monotone.

Thus, the convex optimization problem is equivalent to the fixed point theorem :

$$\underset{x}{\text{minimize}} \quad f(x) \quad \Leftrightarrow \underset{x}{\text{find}} \quad x \in \operatorname{Fix}\mathbb{J}_{\partial f} \quad .$$

Furthermore, $\mathbb{J}_{\partial f}$ is (1/2)-averaged if f is CCP. This is the key stepstone of the reason the fixed point iteration solves the problem.

Theorem 3.15 (Proximal operator)**.** *When f is CCP function and $\alpha > 0$,*

$$\mathbb{J}_{\alpha\partial f}(x) = \underset{y}{\operatorname{argmin}} \left\{ \alpha f(x) + \frac{1}{2}\|x - y\|^2 \right\}.$$

We define this operator as $\operatorname{Prox}_{\alpha f}(y)$*. When* $\operatorname{Prox}_{\alpha f}(y)$ *is efficient to calculate, we call f proximable.*

Theorem 3.16 (Inverse resolvant identity)**.** *When $\mathbb{A}$ is maximally monotone,*

$$\mathbb{J}_{\mathbb{A}} + \mathbb{J}_{\mathbb{A}^{-1}} = \mathbb{I}.$$

As a corollary, when f is CCP,

$$\operatorname{Prox}_f + \operatorname{Prox}_{f^*} = \mathbb{I}.$$

4 Scaled Relative Graph

4.1 Operator Class

Definition 4.1 (Operator class)**.** A set $\mathcal{A}$ is called an *operator class* if it consists of some operators. The operators in $\mathcal{A}$ need not be the operators on the same space. We further define operations on operator class as the following.

- $\mathcal{A}+\mathcal{B} = \{\mathbb{A}+\mathbb{B} : \mathbb{A}\in\mathcal{A}, \mathbb{B}\in\mathcal{B}, \mathbb{A},\mathbb{B} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n, n\in\mathbb{N}\}$.
- $\mathcal{A}\mathcal{B} = \{\mathbb{A}\mathbb{B} : \mathbb{A}\in\mathcal{A}, \mathbb{B}\in\mathcal{B}, \mathbb{A},\mathbb{B} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n, n\in\mathbb{N}\}$.
- $\mathbb{J}_{\mathcal{A}} = \{\mathbb{J}_{\mathbb{A}} : \mathbb{A}\in\mathcal{A}, \mathbb{A} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n, n\in\mathbb{N}\}$.
- $\mathbb{R}_{\mathcal{A}} = \{\mathbb{R}_{\mathbb{A}} : \mathbb{A}\in\mathcal{A}, \mathbb{A} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n, n\in\mathbb{N}\}$.
- $\mathcal{A}^{-1} = \{\mathbb{A}^{-1} : \mathbb{A}\in\mathcal{A}, \mathbb{A} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n, n\in\mathbb{N}\}$.
- $\alpha\mathcal{A} = \{\alpha\mathbb{A} : \mathbb{A}\in\mathcal{A}, \mathbb{A} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n, n\in\mathbb{N}\}$.

Now let's define some useful operator classes.

- $\mathcal{L}_L$: Class of L-Lipschitz operators, $L\in(0,\infty)$.
- $\mathcal{C}_\beta$: Class of β-cocoercive operators, $\beta\in(0,\infty)$.
- $\mathcal{M}$: Class of monotone operators.
- $\mathcal{M}_\mu$: Class of μ-strongly monotone operators, $\mu\in(0,\infty)$.
- $\mathcal{N}_\theta$: Class of θ-averaged operators, $\theta\in(0,1)$. $\mathcal{N}_\theta = (1-\theta)\mathbb{I}+\theta\mathcal{L}_1$.
- $\partial\mathcal{F}_{\mu,L} = \{\partial f : f\in\mathcal{F}_{\mu,L}\}$, $0\le\mu\le L\le\infty$.

The set $\mathcal{F}_{\mu,L}$ is a set of CCP functions such that μ-strongly convex and L-smooth.

4.2 Scaled Relative Graph

Consider values in $\mathbb{R}^n$, x,y,u,v. Define a complex value z as

$$z,\bar{z} = \frac{\|u-v\|}{\|x-y\|}\exp\left[\pm i\angle(u-v,x-y)\right].$$

The angle value $\angle(a,b)$ is defined as

$$\angle(a,b) = \begin{cases} \arccos\left[\frac{\langle a,b\rangle}{\|a\|\|b\|}\right], & a,b\neq 0 \\ 0, & \textit{otherwise.}\end{cases}$$

Now let's define *scaled relative graph* of an operator $\mathbb{A}$.

Definition 4.2 (SRG)**.** The *scaled relative graph* of an operator $\mathbb{A} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is

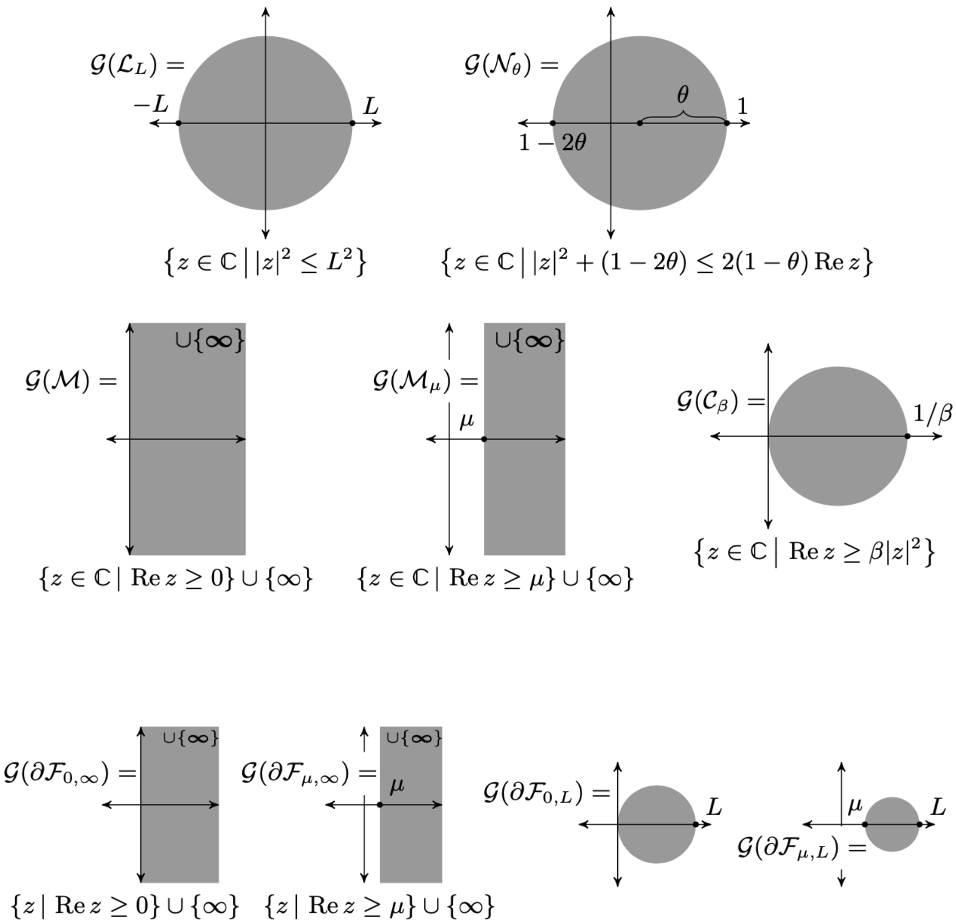
$$\mathcal{G}(\mathbb{A}) = \left\{\frac{\|u-v\|}{\|x-y\|}\exp\left[\pm i\angle(u-v,x-y)\right] : (x,u),(y,v)\in\mathbb{A}, x\neq y\right\}.$$

If $\mathbb{A}$ is multi-valued, we further union the set $\{\infty\}$ to define $\mathcal{G}(\mathbb{A})$.

Definition 4.3 (SRG of an operator class)**.** The *SRG* of an operator class $\mathcal{A}$ is

$$\mathcal{G}(\mathcal{A}) = \cup_{\mathbb{A}\in\mathcal{A}}\mathcal{G}(\mathbb{A}).$$

Here's some examples of SRG graph of operator classes



What we want to do from SRG is to tell whether a given operator is in the operator class using the inclusion of the SRG graphs.

4.3 SRG-full classes

First let's define SRG-full operator class.
Definition 4.4 (SRG-full class)**.** An operator class $\mathcal{A}$ is called *SRG-full* if it satisfies

$$\mathbb{A} \in \mathcal{A} \Leftrightarrow \mathcal{G}(\mathbb{A}) \subseteq \mathcal{G}(\mathcal{A}).$$

Note that $\Leftarrow$ is the key condition on SRG-fullness.

When an operator class $\mathcal{A}$ is SRG-full, we can prove that $\mathbb{A}$ is in $\mathcal{A}$ by observing the inclusion of the SRG graphs. Thus, it is essential to check which operator class is SRG-full.

Theorem 4.5. *An operator class $\mathcal{A}$ is SRG-full if it is defined as*

$$\mathcal{A} = \left\{\mathbb{A} : h\left(\|u-v\|^2, \|x-y\|^2, \langle u-v, x-y\rangle\right) \le 0, \quad \forall (x,u),(y,v) \in \mathbb{A}\right\}$$

for some non-negative homogeneous function $h : \mathbb{R}^3 \to \mathbb{R}$. A function $h : \mathbb{R}^3 \to \mathbb{R}$ is non-negative homogeneous function if

$$h(\theta a, \theta b, \theta c) = \theta h(a,b,c), \quad \forall \theta \ge 0.$$

Proof. Assume $\mathcal{G}(\mathbb{A}) \subseteq \mathcal{G}(\mathcal{A})$. When $(x_A, u_A), (y_A, v_A) \in \mathbb{A}$, there exists z such that

$$\|z\|^2 = \frac{\|u_A - v_A\|^2}{\|x_A - y_A\|^2}, \quad \operatorname{Re} z = \frac{\langle u_A - v_A, x_A - y_A\rangle}{\|x_A - y_A\|^2}.$$

Since $z \in \mathcal{G}(\mathbb{A}) \subseteq \mathcal{G}(\mathcal{A})$, there exists some operator $\mathbb{B} \in \mathcal{A}$ such that

$$\|z\|^2 = \frac{\|u_B - v_B\|^2}{\|x_B - y_B\|^2}, \quad \operatorname{Re} z = \frac{\langle u_B - v_B, x_B - y_B\rangle}{\|x_B - y_B\|^2}$$

is satisfied for some $(x_B, u_B), (y_B, y_B) \in \mathbb{B}$. Since $\mathbb{B} \in \mathcal{A}$, we have

$$h(\|u_B - v_B\|^2, \|x_B - y_B\|^2, \langle u_B - v_B, x_B - y_B\rangle) \le 0.$$

Due to homogeneity,

$$h(\|z\|^2, 1, \operatorname{Re} z) \le 0.$$

Again by homogeneity,

$$h(\|u_A - v_A\|^2, \|x_A - y_A\|^2, \langle u_A - v_A, x_A - y_A\rangle) \le 0,$$

concluding $\mathbb{A} \in \mathcal{A}$. □

As a consequence of this theorem, $\mathcal{L}_L, \mathcal{N}_\theta, \mathcal{M}, \mathcal{M}_\mu, \mathcal{C}_\beta$ is SRG-full operator class. On the other hand, $\partial\mathcal{F}_{\mu,L}$ is not a SRG-full operator since there's some counterexamples :

$$\mathbb{A}(x,y) = (-y, x).$$

Theorem 4.6 (SRG-full preserving operations)**.** *Assume $\mathcal{A}, \mathcal{B}$ are operator classes. Then,*

- $\mathcal{G}(\mathcal{A} \cap \mathcal{B}) \supseteq \mathcal{G}(\mathcal{A}) \cap \mathcal{G}(\mathcal{B})$*. If $\mathcal{A}, \mathcal{B}$ are SRG-full, $\mathcal{A} \cap \mathcal{B}$ is SRG-full and equality holds.*
- $\mathcal{G}(\alpha\mathcal{A}) = \alpha\mathcal{G}(\mathcal{A})$*, $\alpha \ne 0$. If $\mathcal{A}$ is SRG-full, $\alpha\mathcal{A}$ is SRG-full.*
- $\mathcal{G}(\mathbb{I} + \mathcal{A}) = 1 + \mathcal{G}(\mathcal{A})$*. If $\mathcal{A}$ is SRG-full, $\mathbb{I} + \mathcal{A}$ is SRG-full.*
- $\mathcal{G}(\mathcal{A}^{-1}) = \mathcal{G}(\mathcal{A})^{-1}$*. If $\mathcal{A}$ is SRG-full, $\mathcal{A}^{-1}$ is SRG-full.*

Example 4.7 (Convergence analysis of Gradient Descent)**.** Consider the optimization problem

$$\underset{x}{\text{minimize}} \quad f(x)$$

of a μ-strongly convex and L-smooth function f $(0 < \mu < L < \infty)$. The gradient descent method

$$x^{k+1} = x^k - \alpha \nabla f(x^k)$$

converges with rate

$$\|x^k - x^*\| \leq \max(|1 - \alpha\mu|, |1 - \alpha L|)^k \|x^0 - x^*\|,$$

for $\alpha \in (0, 2/L)$.

Example 4.8 (Convergence analysis of Forward Step method-1)**.** Consider the monotone inclusion problem

$$\underset{x}{\text{find}} \quad 0 \in \mathbb{A}x$$

of a μ-strongly monotone and L-Lipschitz operator $\mathbb{A}$ $(0 < \mu < L < \infty)$. The forward step method

$$x^{k+1} = x^k - \alpha \mathbb{A} x^k$$

converges with rate

$$\|x^k - x^*\| \leq (1 - 2\alpha\mu + \alpha^2 L^2)^{k/2} \|x^0 - x^*\|,$$

for $\alpha \in (0, 2\mu/L^2)$.

Example 4.9 (Convergence analysis of Forward Step method-2)**.** Consider the monotone inclusion problem

$$\underset{x}{\text{find}} \quad 0 \in \mathbb{A}x$$

of a μ-strongly monotone and β-cocoersive operator $\mathbb{A}$ $(0 < \mu < 1/\beta < \infty)$. The forward step method

$$x^{k+1} = x^k - \alpha \mathbb{A} x^k$$

converges with rate

$$\|x^k - x^*\| \leq (1 - 2\alpha\mu + \alpha^2 \mu/\beta)^{k/2} \|x^0 - x^*\|,$$

for $\alpha \in (0, 2\beta)$.

Example 4.10 (Convergence analysis of Proximal Point method)**.** Consider the monotone inclusion problem

$$\underset{x}{\text{find}} \quad 0 \in \mathbb{A}x$$

of a maximal μ-strongly monotone operator $\mathbb{A}$. The proximal point method

$$x^{k+1} = \mathbb{J}_{\alpha\mathbb{A}} x^k$$

converges with rate

$$\|x^k - x^*\| \leq \left(\frac{1}{1 + \alpha\mu}\right)^k \|x^0 - x^*\|,$$

for $\alpha > 0$.

5 Fixed Point Iteration

In this section, we cover about *Fixed Point Iteration*, namely *FPI*. The fixed point iteration states that the repeated image via averaged mapping converges to one of the fixed points if such exist.

Theorem 5.1 (Fixed Point Iteration)**.** *Assume $\mathbb{T} : \mathcal{H} \to \mathcal{H}$ is θ-averaged operator on Hilbert space $\mathcal{H}$ with $\theta \in (0, 1)$, and $\operatorname{Fix} \mathbb{T} \neq \emptyset$. Then, the fixed point iteration*

$$x^{k+1} = \mathbb{T}x^k, \quad k = 0, 1, 2, \cdots \tag{FPI}$$

with any starting point x^0 converges to some fixed point :

$$x^k \to x^*, \quad \exists x^* \in \operatorname{Fix} \mathbb{T}.$$

Remark 5.2. The original *Banach Fixed Point Theorem* does not require the existence of $\operatorname{Fix} \mathbb{T}$, since $\operatorname{Fix} \mathbb{T}$ is nonempty when $\mathbb{T}$ is a contraction mapping. However, the nonemptiness of $\operatorname{Fix} \mathbb{T}$ is required if we extend the result to θ-averaged since some θ-averaged has no fixed points.

5.1 Proof of FPI

Proof of FPI. Assume $\mathbb{T}$ is $(1-\theta)\mathbb{I} + \theta\mathbb{C}$ for some nonexpansive mapping $\mathbb{C}$. Then, the FPI is

$$x^{k+1} = (1-\theta)x^k + \theta\mathbb{C}x^k$$

and $\mathbb{C}x^* = x^*$ for all $x^* \in \operatorname{Fix} \mathbb{T}$. From the identity

$$\|(1-\theta)x + \theta y\|^2 = (1-\theta)\|x\|^2 + \theta\|y\|^2 - \theta(1-\theta)\|x-y\|^2,$$

With the nonexpansiveness of $\mathbb{C}$, we have

$$\begin{aligned}
\|x^{k+1} - x^*\|^2 &= (1-\theta)\|x^k - x^*\|^2 + \theta\|\mathbb{C}x^k - x^*\|^2 - \theta(1-\theta)\|\mathbb{C}x^k - x^k\|^2 \\
&\leq (1-\theta)\|x^k - x^*\|^2 + \theta\|x^k - x^*\|^2 - \theta(1-\theta)\|\mathbb{C}x^k - x^k\|^2 \\
&= \|x^k - x^*\|^2 - \frac{1-\theta}{\theta}\|\mathbb{T}x^k - x^k\|^2
\end{aligned}$$

for any $x^* \in \operatorname{Fix} \mathbb{T}$. Note that $\|x^k - x^*\|^2$ and $\|\mathbb{T}x^k - x^k\|^2$ are both nonincreasing sequence. The above inequality gives

$$(k+1)\|x^{k+1} - x^k\|^2 \leq \sum_{j=0}^{k} \|\mathbb{T}x^j - x^j\|^2 \leq \frac{\theta}{1-\theta}\|x^0 - x^*\|^2.$$

Minimization among x^* gives, (since $\operatorname{Fix} \mathbb{T}$ is convex, closed, nonempty set)

$$\|x^{k+1} - x^k\|^2 \leq \frac{1}{k+1}\frac{\theta}{1-\theta}\operatorname{dist}^2(x^0, \operatorname{Fix} \mathbb{T}), \quad x^{k+1} - x^k \to 0.$$

Now let's prove $x^k \to x^*$. From $\|x^{k+1} - x^*\|^2 \leq \|x^0 - x^*\|^2$, $\{x^k\}$ is a sequence in a compact set. Thus, there exists a subsequence $\{x^{k_l}\}$ such that converges $x^{k_l} \to \tilde{x}$ for some $\tilde{x}$. First, $\tilde{x}$ is in $\operatorname{Fix} \mathbb{T}$ since $\mathbb{T} - \mathbb{I}$ is continuous and $\|(\mathbb{T} - \mathbb{I})x^{k_l}\|$ decreasingly converging to 0 gives $(\mathbb{T} - \mathbb{I})\tilde{x} = 0$. Secondly, since we now know that $\tilde{x} \in \operatorname{Fix} \mathbb{T}$, we can say that $\|x^{k+1} - \tilde{x}\| \leq \|x^k - \tilde{x}\|$. Thus, from $\|x^{k_l} - \tilde{x}\| \to 0$, we can conclude that $x^k \to \tilde{x}$. As a conclusion, $x^k \to x^* \in \operatorname{Fix} \mathbb{T}$.

□

Remark 5.3. While the theorem covers θ-averaged mapping, general nonexpansive mapping can be solved by FPI since the fixed point of $\mathbb{C}$ and $(1 - \theta)\mathbb{I} + \theta\mathbb{C}$ coincides.

5.2 Inconsistent Case of FPI

Some could question about the consequence when the existence of fixed point is not ensured. What is the behavior when $\operatorname{Fix} \mathbb{T} = \emptyset$, namely called *Inconsistent*. Such case was studied in the work of A.Pazy(1971), and the result is

$$\lim_{k \to \infty} \frac{x^k}{k} = -\mathbf{v}.$$

where $\mathbf{v}$ is an infimal displacement vector of $\mathbb{T}$. The definition will be discussed later.

Example 5.4. Consider an optimization problem a convex function of

$$f(x) = e^x + x.$$

The gradient descent of this problem takes form of

$$x^{k+1} = x^k - \alpha(e^{x^k} + 1).$$

Which diverges with speed near $-\alpha$ after long iterations.

Proposition 5.5. *Consider an operator $\mathbb{T}$ and $\mathbb{S}$ with a relation of*

$$\mathbb{T} = \mathbb{I} - \theta\mathbb{S}, \quad \theta \in (0, 1).$$

Then, $\mathbb{T}$ is θ-averaged operator if and only if when $\mathbb{S}$ is $(1/2)$-cocoersive operator. Such result also holds for $\theta = 1$, if we consider nonexpansive mapping as θ-averaged with $\theta = 1$.

Example 5.6. The easy way to think of a inconsistent nonexpansive mapping is to think of a (1/2)-cocoersive operator without any zero set. For example, a projection to a closed convex set is 1-cocoersive. If we consider closed convex set C with $0 \neq C$, a θ averaged operator $\mathbb{T} = \mathbb{I} - \Pi_C$ is a (1/2)-averaged operator. What above theorem implies is that fixed point iteration with $\mathbb{T} = \mathbb{I} - \Pi_C$ shows convergence on its normalized iterate $x^k/k \to -\mathbf{v}$ where $\mathbf{v}$ is a minimal norm element of C.

Definition 5.7 (Infimal Element)**.** An *infimal element* $\mathbf{v}$ of nonexpansive $\mathbb{T}$ is defined as the minimal norm element of the closure of the range set : $\overline{\operatorname{range}(\mathbb{I} - \mathbb{T})}$. It is well defined since the set $\operatorname{range}(\mathbb{I} - \mathbb{T})$ is a convex set when $\mathbb{T}$ is nonexpansive.

Theorem 5.8. *Let $\mathbb{T} : \mathcal{H} \to \mathcal{H}$ be nonexpansive mapping. When $\mathbf{v}$ is an infimal displacement vector of $\mathbb{T}$ and $x^0, x^1, x^2, ...$ is a sequence generated by FPI on $\mathbb{T}$, we have strong convergence :*

$$\lim_{k\to\infty} \frac{x^k}{k} = -\mathbf{v}.$$

5.3 Proof of Pazy's Theorem

Lemma 5.9. *Let $C \in \mathcal{H}$ be a closed convex nonempty set and v be a minimal norm element, i.e. $v = \Pi_C 0$. For a sequence $\{u^k\} \subset C$, if $\|u^k\| \to \|v\|$ then*

$$u^k \to v, \quad \|u^k - v\|^2 \le \|u^k\|^2 - \|v\|^2$$

Proof. First, let's prove that $\|v\|^2 \le \langle x, v\rangle$ for all $x \in C$. For any $x \in C$, a point $tx + (1-t)v$ is also in C for all $t \in [0,1]$ due to C's convexity. When we compare the norm, since v is a minimal norm element, we have

$$\|v\|^2 \le \|tx + (1-t)v\|^2 = t^2\|x\|^2 + 2t(1-t)\langle x, v\rangle + (1-t)^2\|v\|^2, \quad \forall t \in (0,1).$$

By subtracting $\|v\|^2$ on both sides and divide by t, we obtain

$$0 \le t\|x\|^2 + 2(1-t)\langle x, v\rangle + (t-2)\|v\|^2, \quad \forall t \in (0,1).$$

By taking a limit $t \to \infty$, we can conclude that $\|v\|^2 \le \langle x, v\rangle$.

Now from this result, since $u^k \in C$ for all k,

$$\|u^k - v\|^2 = \|u^k\|^2 - 2\left\langle u^k, v\right\rangle + \|v\|^2 \le \|u^k\|^2 - \|v\|^2.$$

Thus, if $\|u^k\| \to \|v\|$ then $u^k \to v$. □

Theorem 5.10. *Let $\mathbb{T} : \mathcal{H} \to \mathcal{H}$ nonexpansive operator. Define $d.\tilde{d}$ as*

$$d = \inf\{\|y\| : y \in \overline{\text{range}\,(\mathbb{I} - \mathbb{T})}\}, \quad \tilde{d} = \inf\{\|y\| : y \in \overline{\text{conv}(\text{range}\,(\mathbb{I} - \mathbb{T}))}\},$$

where conv *is a convex hull :* $\text{conv}(A) = \{tx + (1-t)y : t \in [0,1], x, y \in A\}$*. Then,*

$$\tilde{d} \le \liminf_{k\to\infty} \frac{\|\mathbb{T}^k x\|}{k} \le \limsup_{k\to\infty} \frac{\|\mathbb{T}^k x\|}{k} \le d$$

for any choice of x.

Proof. First, to prove the second inequality, consider a sequence z^m such that satisfies $\|z^m - \mathbb{T}z^m\| \to d$. Note that such sequence exists due to the definition of d. Then,

$$\mathbb{T}^k z^m = z^m + \sum_{i=1}^{k} (\mathbb{T}^i - \mathbb{T}^{i-1}) z^m, \quad \|\mathbb{T}^k z^m - \mathbb{T}^k x\| \le \|z^m - x\|.$$

With the help of the triangular inequality,

$$\frac{\|\mathbb{T}^k x\|}{k} \leq \frac{\|z^m\| + \|z^m - x\|}{k} + \|z^m - \mathbb{T}z^m\|.$$

Now take the limit supremum to obtain the desired inequality.

For the first inequality, suppose v is the minimal norm element of the closed convex set $\overline{\text{conv}(\text{range}\,(\mathbb{I}-\mathbb{T}))}$, i.e. $\|v\| = \tilde{d}$. The inner product of v and $\mathbb{T}^k x + kv$ is

$$\left\langle \mathbb{T}^k x + kv, v \right\rangle = \langle x, v\rangle + \sum_{i=1}^{k} \left\langle v - (\mathbb{T}^{i-1}x - \mathbb{T}^i x), v \right\rangle.$$

Since $\mathbb{T}^{i-1}x - \mathbb{T}^i x \in \overline{\text{conv}(\text{range}\,(\mathbb{I}-\mathbb{T}))}$, we have $\left\langle v - (\mathbb{T}^{i-1}x - \mathbb{T}^i x), v \right\rangle \leq 0$. Thus,

$$\left\langle \mathbb{T}^k x + kv, v \right\rangle \leq \langle x, v\rangle$$

and as a consequence, if $\|v\| = \tilde{d} \neq 0$,

$$\frac{\|\mathbb{T}^k x\|}{k} \geq -\frac{\left\langle \mathbb{T}^k x, v \right\rangle}{k\|v\|} \geq \|v\| - \frac{\langle x, v\rangle}{k\|v\|}.$$

Take the limit to conclude the first inequality. The case of $\tilde{d} = 0$ is trivial. □

Remark 5.11. For the remark, when the $\mathbb{T}$ is nonexpansive operator on $\mathcal{H}$, range $(\mathbb{I}-\mathbb{T})$ is always convex set. Thus, the above theorem gives the Pazy's result straightforward since $d = \tilde{d}$. However, the above theorem also works when $\mathbb{T}$ is only nonexpansive on elements in closed set D. In this case, range $(\mathbb{I}-\mathbb{T})$ need not be convex.

proof of Pazy's Theorem. Due to convexity of range $(\mathbb{I}-\mathbb{T})$,

$$-\frac{\mathbb{T}^k x}{k} + \frac{x}{k} = \frac{1}{k}\sum_{i=1}^{k}(\mathbb{I}-\mathbb{T})\mathbb{T}^{i-1}x \in \overline{\text{range}\,(\mathbb{I}-\mathbb{T})}.$$

Since $\left\|-\frac{\mathbb{T}^k x}{k} + \frac{x}{k}\right\| \to \|\mathbf{v}\|$, we can conclude that $-\frac{\mathbb{T}^k x}{k} \to \mathbf{v}$ and $k \to \infty$. □

Remark 5.12. It is further known that the difference of the iterates also converges :

$$x^{k+1} - x^k \to -\mathbf{v}.$$

Example 5.13. The inconsistent case of FPI can be useful in measuring the minimal norm or distance between two closed convex sets. For an example, consider an optimization problem

$$\underset{x}{\text{minimize}} \quad f(x) + g(x) \quad , \quad f = \delta_A, g = \delta_B,$$

where A, B are closed nonempty convex sets with $A \cap B = \emptyset$. One can notice that the problem is infeasible since the domain where the function value exists is an empty set. However, we can still apply DRS method for convex sets. The DRS method will be explained in the next remark. When DRS method is applied, the iteration is :

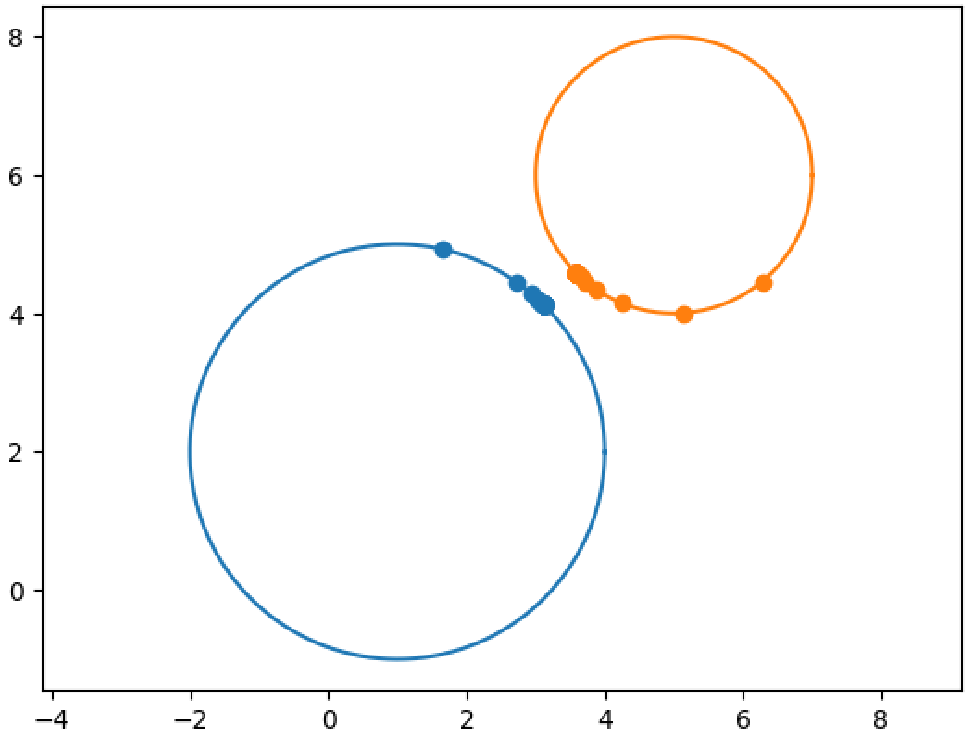
$$\begin{aligned} x^{k+1/2} &= \mathbb{J}_{\partial g}(z^k) = \Pi_B(z^k) \\ x^{k+1} &= \mathbb{J}_{\partial f}(2x^{k+1/2} - z^k) = \Pi_A(2x^{k+1/2} - z^k) \\ z^{k+1} &= z^k + x^{k+1} - x^{k+1/2}. \end{aligned}$$

Such iteration converges as

$$x^{k+1/2} \to x_B \in B, \quad x^{k+1} \to x_A \in A, \quad \frac{z^k}{k} \to v = x_A - x_B$$

where $\|v\| = \text{dist}(A, B)$.

The following figure depicts x^k and $x^{k+1/2}$ when the sets A, B are circles.



6 Randomized Method

In practical issues, randomization technique is broadly used in the area of optimization and machine learning. Such technique gives advantages such as faster convergence speed. In this chapter we will cover about randomized version of FPI, namely called *RC-FPI* or *Randomized (block) Coordinate update Fixed Point Iteration.* The simplest version of RC-FPI by $\mathbb{T} : \mathbb{R}^m \to \mathbb{R}^m$ is a single coordinate update with uniform selection,

$$x^{k+1} = \mathbb{T}_{i^k} x^k, \quad (\mathbb{T}_{i^k} x)_j = \begin{cases} x & j \neq i^k \\ (\mathbb{T}x)_{i^k} & j = i^k, \end{cases}$$

where i^k are selected with IID on uniform distribution of $1, 2, \cdots, m$. Now let's generalize this notion on the Hilbert space.

First, let's clarify the underlying space. The underlying space is a real Hilbert space $\mathcal{H}$, which is consisted of m real Hilbert spaces.

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots \mathcal{H}_m.$$

An element $u \in \mathcal{H}$ can be decomposed into m blocks as

$$u = (u_1, u_2, \ldots, u_m), \quad u_i \in \mathcal{H}_i,$$

and u_i is called the ith block coordinates of u.

The Hilbert space $\mathcal{H}$ has its induced norm and inner product as

$$\|x\|^2 = \sum_{i=1}^{m} \|x_i\|_i^2, \qquad \langle x, y \rangle = \sum_{i=1}^{m} \langle x_i, y_i \rangle_i,$$

for all $x, y \in \mathcal{H}$, where $\|\cdot\|_i$ and $\langle \cdot, \cdot \rangle_i$ are the norm and inner product of $\mathcal{H}_i$ and x_i, y_i are ith block coordinates of x, y, respectively.

Consider a linear, bounded, self-adjoint and positive definite operator $M : \mathcal{H} \to \mathcal{H}$. The M-norm and M-inner product of $\mathcal{H}$ are defined as

$$\|x\|_M = \sqrt{\langle x, Mx \rangle}, \qquad \langle x, y \rangle_M = \langle x, My \rangle,$$

which can also be a pair of norm and inner product of the space $\mathcal{H}$. $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ are simply the instances of M-norm and M-inner product with M as an identity map. For the remark, the map M can be expressed as a symmetric positive definite matrix if $\mathcal{H} = \mathbb{R}^n$. In this case, M-inner product and M-norm are

$$\|x\|_M = \sqrt{x^T M x}, \qquad \langle x, y \rangle_M = x^T M y.$$

Define the M-variance of a random variable X with the domain $\mathcal{H}$ as

$$\mathrm{Var}_M[X] = \mathbb{E}[\|X\|_M^2] - \|\mathbb{E}[X]\|_M^2.$$

Consider a θ-averaged operator $\mathbb{T}\colon \mathcal{H} \to \mathcal{H}$ with its corresponding (1/2)-cocoercive operator $\mathbb{S} = \theta^{-1}(\mathbb{I} - \mathbb{T})$ with $\theta \in (0, 1]$. To clarify, we will refer to non-expansive operators as θ-averaged operators with $\theta = 1$. Define $\mathbb{S}_i\colon \mathcal{H} \to \mathcal{H}$ for $i = 1, 2, \ldots, m$ as $\mathbb{S}_i x = (0, \ldots, 0, (\mathbb{S}x)_i, 0, \ldots, 0)$, where $(\mathbb{S}x)_i \in \mathcal{H}_i$.

We call $\mathcal{I} = (\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_m) \in [0, 1]^m \subset \mathbb{R}^m$ a *selection vector* and use it as follows. Define $\mathbb{S}_{\mathcal{I}}\colon \mathcal{H} \to \mathcal{H}$ and $\mathbb{T}_{\mathcal{I}}\colon \mathcal{H} \to \mathcal{H}$ as

$$\mathbb{S}_{\mathcal{I}} = \sum_{i=1}^{m} \mathcal{I}_i \mathbb{S}_i, \qquad \mathbb{T}_{\mathcal{I}} = \mathbb{I} - \theta \mathbb{S}_{\mathcal{I}}.$$

We can think of $\mathbb{S}_{\mathcal{I}}$ as the selection of blocks based on $\mathcal{I}$ and $\mathbb{T}_{\mathcal{I}}$ as the update based on the selected blocks. Throughout this paper, we assume that $\mathcal{I}$ is randomly sampled from a distribution on $[0, 1]^m$ that satisfies the *uniform expected step-size condition*

$$\mathbb{E}_{\mathcal{I}}\left[\mathcal{I}\right] = \alpha \mathbf{1} \tag{1}$$

for some $\alpha \in (0, 1]$, where $\mathbf{1} \in \mathbb{R}^m$ is the vector with all entries equal to 1. (Note, $\mathcal{I} \in [0, 1]^m$ already implies $\alpha \in [0, 1]$ so we are additionally assuming that $\alpha > 0$.) The randomized coordinate fixed-point iteration (RC-FPI) is defined as

$$x^{k+1} = \mathbb{T}_{\mathcal{I}^k} x^k, \qquad k = 0, 1, 2, \ldots, \tag{RC-FPI}$$

where $\mathcal{I}^0, \mathcal{I}^1, \ldots$ is sampled IID and $x^0 \in \mathcal{H}$ is a starting point.

(RC-FPI) is a randomized variant of (FPI). The uniform expected step-size condition (1) allows us to view one step of (RC-FPI) to be corresponding to a step of (FPI) with $\bar{\mathbb{T}}\colon \mathcal{H} \to \mathcal{H}$ defined as

$$\bar{\mathbb{T}} x = \mathbb{E}_{\mathcal{I}}\left[\mathbb{T}_{\mathcal{I}} x\right], \qquad \forall x \in \mathcal{H}.$$

Equivalently, $\bar{\mathbb{T}} = \mathbb{I} - \alpha\theta\mathbb{S}$. As a consequence, $\bar{\mathbb{T}}$ is also a $\alpha\theta$-averaged operator.

Before moving on, let's define β value embedded by the probability distribution of the selection vector $\mathcal{I}$. For any $u \in \mathcal{H}$ and selection vector $\mathcal{I}$, define

$$u_{\mathcal{I}} = \sum_{i=1}^{m} \underbrace{\mathcal{I}_i}_{\in \mathbb{R}} \underbrace{(0, \ldots, 0, u_i, 0, \ldots, 0)}_{\in \mathcal{H}},$$

where $u_i \in \mathcal{H}_i$ for $i = 1, \ldots, m$. If $\mathcal{I}$ satisfies the uniform expected step-size condition (1) with $\alpha \in (0, 1]$, then clearly $\mathbb{E}_{\mathcal{I}}\left[u_{\mathcal{I}}\right] = \alpha u$. Let $\beta > 0$ be a coefficient such that

$$\mathbb{E}_{\mathcal{I}}\left[\left\|u_{\mathcal{I}}\right\|_M^2\right] \le \beta \left\|u\right\|_M^2, \qquad \forall\, u \in \mathcal{H}. \tag{2}$$

For the remark, β is necessarily greater or equal to α^2.

Lemma 6.1. *Consider a Hilbert space $\mathcal{H}$ with its norm $\|\cdot\|$. If $\mathcal{I}$ satisfies the uniform expected step-size condition (1) with $\alpha \in (0, 1]$, then $\beta = \alpha$ satisfies (2).*

6.1 Convergence of RC-FPI

Randomization may accelerate, but it is not useful until the convergence is guaranteed. Thankfully, with (RC-FPI) also converges to the fixed point even with probability 1.

Theorem 6.2 (Convergence of RC-FPI)**.** *Assume $\mathbb{T} : \mathcal{H} \to \mathcal{H}$ as a θ-averaged operator on separable Hilbert space $\mathcal{H}$ with $\theta \in (0,1)$ with $\operatorname{Fix} \mathbb{T} \neq \emptyset$. When $\alpha > \theta\beta$, (RC-FPI) by $\mathbb{T}$ with condition (1) converges to one of the fixed point with probability 1 :*

$$x^k \overset{\text{a.s.}}{\to} x^*,$$

for some choice of $x^* \in \operatorname{Fix} \mathbb{T}$. Notation $\overset{\text{a.s.}}{\to}$ denotes a strong convergence of Hilbert space with probability of 1.

Remark 6.3. Results in (RC-FPI) is is fairly non-trivial. For the remark, a randomized operator $\mathbb{T}_{\mathcal{I}}$ may not be non-expansive even when $\mathbb{T}$ is. For an example, consider a rotation operator.

Remark 6.4. Note that in $\|\cdot\|$, $\theta\beta < \alpha$ is satisfied with the choice of $\beta = \alpha$.

Short proof. Take a conditional expectation at step k on $\left\|x^{k+1} - x^*\right\|_M^2$ for $x^* \in \operatorname{Fix} \mathbb{T}$.

$$\begin{aligned}
&\mathbb{E}\left[\left\|x^{k+1} - x^*\right\|_M^2 \middle| \mathcal{F}_k\right] \\
&\quad = \left\|x^k - x^*\right\|_M^2 - 2\theta \left\langle \mathbb{E}\left[\mathbb{S}_{\mathcal{I}^k} x^k \middle| \mathcal{F}_k\right], x^k - x^* \right\rangle_M + \theta^2 \mathbb{E}\left[\left\|\mathbb{S}_{\mathcal{I}^k} x^k\right\|_M^2 \middle| \mathcal{F}_k\right] \\
&\quad = \left\|x^k - x^*\right\|_M^2 - 2\theta\alpha \left\langle \mathbb{S}x^k - \mathbb{S}x^*, x^k - x^* \right\rangle_M + \theta^2 \beta \left\|\mathbb{S}x^k\right\|_M^2 \\
&\quad \leq \left\|x^k - x^*\right\|_M^2 - \theta(\alpha - \theta\beta) \left\|\mathbb{S}x^k\right\|_M^2 .
\end{aligned}$$

By supermartingale convergence theorem, for each $x^* \in \operatorname{Fix} \mathbb{T}$,

$$\sum_0^\infty \left\|\mathbb{S}x^k\right\|_M^2 < \infty, \quad \lim_{k\to\infty} \left\|x^k - x^*\right\|_M^2 \text{ exists}$$

with probability 1. Since $\mathcal{H}$ is separable, we can say with probability 1 that

$$\left\|\mathbb{S}x^k\right\|_M \to 0, \quad \lim_{k\to\infty} \left\|x^k - x^*\right\|_M^2 \text{ exists}$$

for all $x^* \in \operatorname{Fix} \mathbb{T}$. Also note that $\mathbb{E}\left\|x^k - x^*\right\|_M^2$ is nonincreasing sequence. With the same arguments from the proof of FPI with the notion of with probability 1 appended, we can conclude the proof. In detail, with probability 1, sequence x^k is bounded for a sufficiently large k and thus has a converging subsequence. Such subsequence converges to the value in $\operatorname{Fix} \mathbb{T}$ since $\mathbb{S}x^k \to 0$. Since $\lim \left\|x^k - \tilde{x}\right\|_M^2$ exists and $\lim \left\|x^{k_l} - \tilde{x}\right\|_M^2 = 0$, we can conclude that $x^k \to \tilde{x} \in \operatorname{Fix} \mathbb{T}$. □

Remark 6.5. Note that the separability of $\mathcal{H}$ (i.e. $\mathcal{H}$ contains countable, dense subset) is required. It is important since x^* that bounds the sequence and $\tilde{x}$, the convergence value of the subsequence x^{k_l} may differ.

6.2 RC-FPI admits faster convergence

At glance, the form of (RC-FPI) doesn't seem faster compared to the original FPI. It is true that (RC-FPI) is not fast for all types of problems. (RC-FPI) takes advantages when the problem, or the operator, is coordinate friendly.

Definition 6.6. An operator is called *Coordinate friendly* if it satisfies

$$\mathcal{F}\left[x \mapsto (\mathbb{T}x)_i\right] \leq \frac{C}{m}\mathcal{F}\left[x \mapsto \mathbb{T}x\right], \quad \forall i = 1, 2, \cdots, m$$

for some not too large $C > 0$. $\mathcal{F}[x \mapsto z]$ denotes computational cost of calculating z from x. Many optimization problems of the form

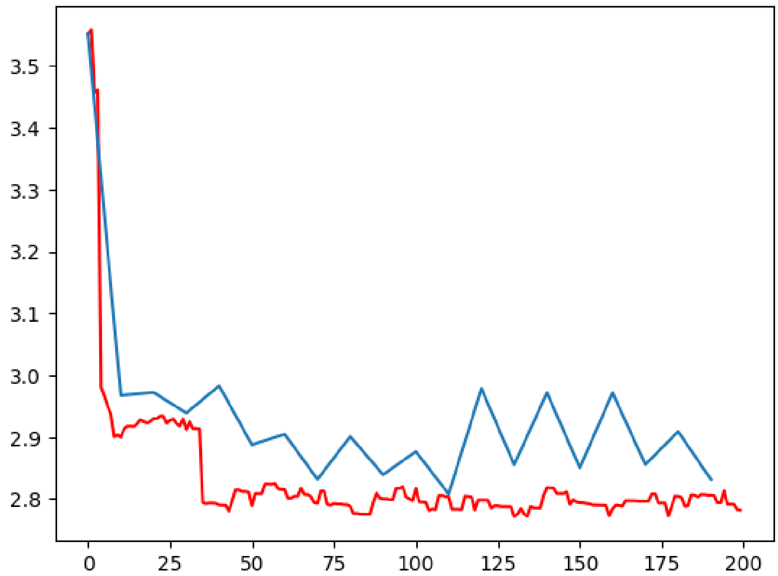
$$\underset{x_1, x_2, \cdots, x_m}{\text{minimize}} \quad \sum_{i=1}^{m} f_i(x_i)$$

can be solved by coordinate friendly operator.

Example 6.7 (LASSO)**.** A famous example is the LASSO optimization. The LASSO problem has a form of

$$\underset{x}{\text{minimize}} \quad L(x) \quad , \quad L(x) = \frac{1}{2}\|Xx - y\|_2^2 + \lambda \|x\|_1.$$

We can easily check that the gradient of the $L(x)$ can be computed in each index independently.



7 Inconsistent case of RC-FPI

In this section, I introduce my work on the behavior of (RC-FPI) in the general case, including when the fixed point doesn't exist. In this seminar, the key concepts of the proofs are provided instead of the full proof since it is quite technical. The full proof can be checked in the paper arXiv:2305.12211.

Following are key results. First, (RC-FPI) also obtains similar behavior of the Pazy's theorem. The normalized iterate x^k/k converges to $-\alpha\mathbf{v}$ both in L^2 and almost surely.

$$\frac{x^k}{k} \xrightarrow{L^2} -\alpha\mathbf{v}, \quad \frac{x^k}{k} \xrightarrow{\text{a.s.}} -\alpha\mathbf{v}.$$

$\mathbf{v}$ is the infimal displacement vector of $\mathbb{T}$, or equivalently $\alpha\mathbf{v}$ is the infimal displacement vector of $\bar{\mathbb{T}}$.

Second, it is possible to show that

$$\limsup_{k\to\infty} k\mathrm{Var}_M\left[\frac{x^k}{k}\right] \le (\beta-\alpha^2)\left\|\mathbf{v}\right\|_M^2,$$

an analogous result of central limit theorem.

Before moving on to the main theorems and proofs, here's the one-step inequality which will be the key inequality of each proofs.

Lemma 7.1. *Let $\mathbb{T}\colon \mathcal{H}\to\mathcal{H}$ be θ-averaged respect to $\|\cdot\|_M$ with $\theta\in(0,1]$. Let $\mathcal{I}$ be a random selection vector with distribution satisfying the uniform expected step-size condition (1) with $\alpha\in(0,1]$. Assume (2) holds with some β. For any $x,z\in\mathcal{H}$,*

$$\mathop{\mathbb{E}}_{\mathcal{I}}\left[\left\|\mathbb{T}_{\mathcal{I}}x-\bar{\mathbb{T}}z\right\|_M^2\right] \le \|x-z\|_M^2 + \theta^2\left(\beta-\alpha^2\right)\|\mathbb{S}x\|_M^2 - \alpha\theta\left(1-\alpha\theta\right)\|\mathbb{S}x-\mathbb{S}z\|_M^2.$$

Proof. First, substitute $\mathbb{T}_{\mathcal{I}} = \mathbb{I}-\theta\mathbb{S}_{\mathcal{I}}$ and $\bar{\mathbb{T}} = \mathbb{I}-\alpha\theta\mathbb{S}$ at $\mathbb{E}_{\mathcal{I}}\left[\left\|\mathbb{T}_{\mathcal{I}}x-\bar{\mathbb{T}}z\right\|_M^2\right]$.

$$\begin{aligned}\mathbb{E}\left[\left\|\mathbb{T}_{\mathcal{I}}x-\bar{\mathbb{T}}z\right\|_M^2\right] &= \mathbb{E}\left[\left\|x-z-\theta\left(\mathbb{S}_{\mathcal{I}}x-\alpha\mathbb{S}z\right)\right\|_M^2\right]\\ &= \|x-z\|_M^2+\theta^2\mathbb{E}\left[\left\|\mathbb{S}_{\mathcal{I}}x-\alpha\mathbb{S}z\right\|_M^2\right]-2\alpha\theta\left\langle x-z,\mathbb{S}x-\mathbb{S}z\right\rangle_M\\ &\le \|x-z\|_M^2+\theta^2\mathbb{E}\left[\left\|\mathbb{S}_{\mathcal{I}}x-\alpha\mathbb{S}z\right\|_M^2\right]-\alpha\theta\left\|\mathbb{S}x-\mathbb{S}z\right\|_M^2.\end{aligned}$$

The (1/2)-cocoercive property of the operator $\mathbb{S}$ is used to obtain the last inequality. Since $\mathbb{E}[\mathbb{S}_{\mathcal{I}}x]=\alpha\mathbb{S}x$, the second term can be bounded as

$$\begin{aligned}\mathbb{E}\left[\left\|\mathbb{S}_{\mathcal{I}}x-\alpha\mathbb{S}z\right\|_M^2\right] &= \mathbb{E}\left[\left\|\left(\mathbb{S}_{\mathcal{I}}x-\alpha\mathbb{S}x\right)+\alpha\left(\mathbb{S}x-\mathbb{S}z\right)\right\|_M^2\right]\\ &= \mathrm{Var}_M\left[\mathbb{S}_{\mathcal{I}}x\right]+\alpha^2\left\|\mathbb{S}x-\mathbb{S}z\right\|_M^2\\ &\le \left(\beta-\alpha^2\right)\|\mathbb{S}x\|_M^2+\alpha^2\left\|\mathbb{S}x-\mathbb{S}z\right\|_M^2,\end{aligned}$$

and we can get the desired inequality. □

As one can see from this lemma, key idea of the main theorem's proofs is comparing the random sequence by (RC-FPI) with the fixed point iteration by $\bar{\mathbb{T}}$. Define a deterministic sequence $z^0, z^1, z^2, \ldots$ as

$$z^{k+1} = \bar{\mathbb{T}} z^k, \qquad k = 0, 1, 2, \ldots. \tag{FPI with $\bar{\mathbb{T}}$}$$

Throughout this section, x^k will usually denote a random sequence by (RC-FPI) via $\mathbb{T}$ and z^k will be referred as a sequence (FPI with $\bar{\mathbb{T}}$).

7.1 L^2 convergence of the normalized iterate

Theorem 7.2. *Let $\mathbb{T}\colon \mathcal{H} \to \mathcal{H}$ be θ-averaged with respect to $\|\cdot\|$-norm with $\theta \in (0,1]$. Assume $\mathcal{I}^0, \mathcal{I}^1, \ldots$ is sampled IID from a distribution satisfying the uniform expected step-size condition (1) with $\alpha \in (0,1]$. Let $x^0, x^1, x^2, \ldots$ be the iterates of (RC-FPI). Then*

$$\frac{x^k}{k} \overset{L^2}{\to} -\alpha \mathbf{v}$$

as $k \to \infty$, where $\mathbf{v}$ is the infimal displacement vector of $\mathbb{T}$.

Proof. From the one-step inequality Lemma 7.1, by taking full expectation

$$\begin{aligned}
&\mathbb{E}\left[\left\|x^{k+1} - z^{k+1}\right\|_M^2\right] - \mathbb{E}\left[\left\|x^k - z^k\right\|_M^2\right] \\
&\quad\leq \theta^2\left(\beta - \alpha^2\right)\mathbb{E}[\|\mathbb{S}x\|_M^2] - \alpha\theta\left(1 - \alpha\theta\right)\mathbb{E}[\|\mathbb{S}x - \mathbb{S}z\|_M^2] \\
&\quad= -\theta\left(\alpha - \beta\theta\right)\mathbb{E}[\|\mathbb{S}x\|_M^2] + 2\alpha\theta\left(1 - \alpha\theta\right)\langle \mathbb{E}[\mathbb{S}x], \mathbb{S}z\rangle_M - \alpha\theta\left(1 - \alpha\theta\right)\|\mathbb{S}z\|_M^2 \\
&\quad\leq -\theta^{-1}\alpha\left(1 - \alpha\theta\right)\|\mathbf{v}\|_M^2 + 2\alpha\theta\left(1 - \alpha\theta\right)\|\mathbb{E}[\mathbb{S}x]\|_M \|\mathbb{S}z\|_M,
\end{aligned}$$

where the last inequality is from $\mathbf{v}$ being infimal displacement vector, which implies $\|\mathbf{v}\|_M \leq \|\theta\mathbb{S}x\|_M, \|\theta\mathbb{S}z\|_M$. If we can bound $\|\mathbb{E}[\mathbb{S}x]\|_M$ and $\|\mathbb{S}z\|_M$ (which is possible from following two lemmas), we can conclude the proof since

$$\mathbb{E}\left[\left\|x^{k+1} - z^{k+1}\right\|_M^2\right] \leq \mathbb{E}\left[\left\|x^k - z^k\right\|_M^2\right] + A \leq kA + C,$$

and dividing each side with $(k+1)^2$ gives L^2 distance converging to 0. □

Lemma 7.3. *$\mathbb{T}\colon \mathcal{H} \to \mathcal{H}$ is a θ-averaged with $\theta \in (0,1]$ and choose any starting point $z^0 \in \mathcal{H}$ for (FPI with $\bar{\mathbb{T}}$). When $\mathbb{S} = \theta^{-1}(\mathbb{I} - \mathbb{T})$,*

$$\left\|\mathbb{S}z^k\right\|_M \leq \left\|\mathbb{S}z^{k-1}\right\|_M \leq \cdots \leq \left\|\mathbb{S}z^0\right\|_M.$$

Proof of Lemma 7.3. From $\mathbb{S}$ being $(1/2)$-cocoercive operator, we can check that

$$\theta\left\langle \mathbb{S}\mathbb{T}z - \mathbb{S}z, -\mathbb{S}z - \mathbb{S}\mathbb{T}z\right\rangle_M \geq (1 - \theta)\left\|\mathbb{S}\mathbb{T}z - \mathbb{S}z\right\|_M^2 \geq 0,$$

which is equivalent to

$$\|\mathbb{S}\mathbb{T}z\|_M^2 \leq \|\mathbb{S}z\|_M^2.$$

□

Lemma 7.4. *$\mathbb{T}\colon \mathcal{H} \to \mathcal{H}$ is a θ-averaged with $\theta \in (0,1]$ and choose any starting point $x^0 \in \mathcal{H}$ for (RC-FPI). When $\mathbb{S} = \theta^{-1}(\mathbb{I} - \mathbb{T})$,*

$$\left\|\mathbb{E}\left[\mathbb{S}\mathbb{T}_{\mathcal{I}^k}\dots\mathbb{T}_{\mathcal{I}^0}x^0\right]\right\|_M \le \beta^{1/2}\alpha^{-1}\left\|\mathbb{S}x^0\right\|_M$$

holds if $\mathcal{I}^0,,\mathcal{I}^1,\dots,\mathcal{I}^k$ follow IID distribution with the condition (1) with $\alpha \in (0,1]$ and (2) holds with some β that $\beta \le \alpha/\theta$.

Proof of Lemma 7.4. We can easily check that from (1/2)-cocoersivity

$$\mathbb{E}\left[\|\mathbb{T}_{\mathcal{I}}X - \mathbb{T}_{\mathcal{I}}Y\|_M^2\right] \le \mathbb{E}\left[\|X - Y\|_M^2\right].$$

First apply above inequality repeatedly, then apply Jensen's Inequality on the LHS, and finally set up X, Y as $X = \mathbb{T}_{\mathcal{I}^0}x^0$ and $Y = x^0$. Then as a result, we have an inequality

$$\left\|\mathbb{E}\left[\mathbb{T}_{\mathcal{I}^k}\dots\mathbb{T}_{\mathcal{I}^1}\mathbb{T}_{\mathcal{I}^0}x^0 - \mathbb{T}_{\mathcal{I}^k}\dots\mathbb{T}_{\mathcal{I}^1}x^0\right]\right\|_M^2 \le \mathbb{E}\left[\left\|\theta\mathbb{S}_{\mathcal{I}^0}x^0\right\|_M^2\right] \le \beta\left\|\theta\mathbb{S}x^0\right\|_M^2.$$

Since $\mathcal{I}_0, \mathcal{I}_1, \dots, \mathcal{I}_n$ are IID, the following equivalence holds and ends the proof.

$$\mathbb{E}\left[\mathbb{T}_{\mathcal{I}^k}\dots\mathbb{T}_{\mathcal{I}^1}x^0\right] = \mathbb{E}\left[\mathbb{T}_{\mathcal{I}^{k-1}}\dots\mathbb{T}_{\mathcal{I}^0}x^0\right].$$

□

7.2 Almost sure convergence of normalized iterate

Theorem 7.5. *Under the conditions of Theorem 7.2 with $\theta \in (0,1)$, x^k/k is strongly convergent to $-\alpha\mathbf{v}$ in probability 1. In other words,*

$$\frac{x^k}{k} \overset{\text{a.s.}}{\to} -\alpha\mathbf{v}$$

as $k \to \infty$.

Proof. To use the Robbins-Siegmund quasi-martingale theorem Lemma 7.6, we cannot take full expectation to bound the extra terms in Lemma 7.1. Here, we provide alternate way to bound the last two terms in Lemma 7.1.

$$\begin{aligned} &-\alpha\theta(1-\alpha\theta)\|\mathbb{S}x - \mathbb{S}z\|_M^2 + \theta^2\left(\beta - \alpha^2\right)\|\mathbb{S}x\|_M^2 \\ &\quad = -\theta(\alpha - \beta\theta)\left\|\mathbb{S}x - \frac{\alpha - \alpha^2\theta}{\alpha - \beta\theta}\mathbb{S}z\right\|_M^2 + \underbrace{\frac{\alpha\theta^2(1-\alpha\theta)(\beta-\alpha^2)}{\alpha - \beta\theta}}_{=:B\ge 0}\|\mathbb{S}z\|_M^2. \end{aligned}$$

From Lemma 7.1,

$$\mathbb{E}_{\mathcal{I}^k}\left[\left\|\frac{x^k}{k} - \frac{z^k}{k}\right\|_M^2 \mid \mathcal{F}_{k-1}\right] \le \left\|\frac{x^{k-1}}{k-1} - \frac{z^{k-1}}{k-1}\right\|_M^2 + \frac{B}{k^2}\left\|\mathbb{S}z^{k-1}\right\|_M^2,$$

where $x^0, x^1, x^2, \dots$ is a random sequence generated by (RC-FPI) with $\mathbb{T}$, $z^0, z^1, z^2, \dots$ is a sequence generated by (FPI with $\bar{\mathbb{T}}$) and starting point $z^0 = x^0$, and $\mathcal{F}_k$ is a filtration consisting of information up to nth iteration. Recall that $\left\|\mathbb{S}z^{k-1}\right\|_M^2 \le \left\|\mathbb{S}z^0\right\|_M^2$. Thus, from the quasi-martingale theorem, the random sequence $\left\|\frac{x^k}{k} - \frac{z^k}{k}\right\|_M^2$ converges almost surely to some random variable. Then, by Fatou's lemma and the L^2 convergence of Theorem 7.2, we have

$$\mathbb{E}\left[\lim_{k\to\infty}\left\|\frac{x^k}{k} - \frac{z^k}{k}\right\|_M^2\right] \le \lim_{k\to\infty}\mathbb{E}\left[\left\|\frac{x^k}{k} - \frac{z^k}{k}\right\|_M^2\right] = 0.$$

Thus, as $k \to \infty$,

$$\left\|\frac{x^k}{k} - \frac{z^k}{k}\right\|_M^2 \overset{\text{a.s.}}{\to} 0,$$

and appealing to Pazy's result, we conclude the almost sure convergence. □

For the remark, here's full statement of Robbins-Siegmund quasi-martingale theorem.

Lemma 7.6. *$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots$ is a sequence of sub-σ-algebras of $\mathcal{F}$ where $(\Omega, \mathcal{F}, P)$ is a probability space. When $X_k, b_k, \tau_k, \zeta_k$ are non-negative $\mathcal{F}_k$-random variables such that*

$$\mathbb{E}\left[X_{k+1} \mid \mathcal{F}_k\right] \le \left(1 + b_k\right) X_k + \tau_k - \zeta_k,$$

$\lim_{k\to\infty} X_k$ exists and is finite and $\sum_{k=1}^{\infty} \zeta_k < \infty$ almost surely if $\sum_{k=1}^{\infty} b_k < \infty, \sum_{k=1}^{\infty} \tau_k < \infty$.

7.3 Bias and variance of normalized iterates

Theorem 7.7. *Let $\mathbb{T}\colon \mathcal{H} \to \mathcal{H}$ be θ-averaged with respect to $\|\cdot\|_M$ with $\theta \in (0, 1]$. Let $\mathbf{v}$ be the infimal displacement vector of $\mathbb{T}$. Assume $\mathcal{I}^0, \mathcal{I}^1, \dots$ is sampled IID from a distribution satisfying the uniform expected step-size condition (1) with $\alpha \in (0, 1]$, and assume (2) holds with some $\beta > 0$ such that $\beta < \alpha/\theta$. Let $x^0, x^1, x^2, \dots$ be the iterates of (RC-FPI).*

(a) If $\mathbf{v} \in \operatorname{range}(\mathbb{I} - \mathbb{T})$, then as $k \to \infty$,

$$\mathbb{E}\left[\left\|\frac{x^k}{k} + \alpha\mathbf{v}\right\|_M^2\right] \lesssim \frac{\left(\beta - \alpha^2\right)\|\mathbf{v}\|_M^2}{k}.$$

(b) In general, regardless of whether $\mathbf{v}$ is in $\operatorname{range}(\mathbb{I} - \mathbb{T})$ or not,

$$\operatorname{Var}_M\left(\frac{x^k}{k}\right) \lesssim \frac{\left(\beta - \alpha^2\right)\|\mathbf{v}\|_M^2}{k}$$

as $k \to \infty$.

To clarify, the precise meaning of the first asymptotic statement of (a) is

$$\limsup_{k\to\infty} k\,\mathbb{E}\left[\left\|\frac{x^k}{k}+\alpha\mathbf{v}\right\|_M^2\right] \le (\beta-\alpha^2)\,\|\mathbf{v}\|_M^2 .$$

The precise meaning of the asymptotic statement of (b) is defined similarly.

In this seminar, detailed calculation is omitted. The full proofs and omitted proofs of lemmas are available in my paper. Let's start the proof.

Proof. Let $z^0, z^1, z^2, \ldots$ be the iterates of (FPI with $\bar{\mathbb{T}}$) with z^0 satisfying $\theta\mathbb{S}z^0 = \mathbf{v}$. Then, $\theta\mathbb{S}z^k = \mathbf{v}$ for all $k \in \mathbb{N}$. Apply Lemma 7.1 on x^k and z^k and take full expectation to get

$$\mathbb{E}\left[k\left\|\frac{x^k}{k}-\frac{z^k}{k}\right\|_M^2\right] \le \frac{1}{k}\left\|x^0-z^0\right\|_M^2 + \mathbb{E}\left[\frac{1}{k}\sum_{j=0}^{k-1}U^j\right]$$

where $U^0, U^1, U^2, \ldots$ is a sequence of random variables :

$$U^k = -\,\alpha\left(\theta^{-1}-\alpha\right)\left\|\theta\mathbb{S}x^k-\mathbf{v}\right\|_M^2 + \theta^2\left(\beta-\alpha^2\right)\left\|\mathbb{S}x^k\right\|_M^2 .$$

The key of this proof is to bound U^k asymptotically as the right hand side of the theorem. Then due to Cesàro mean we may conclude the proof. We will approach such bound using $\mathbb{S}x^k$ and $\theta\mathbb{S}x^k - \mathbf{v}$ being nearly orthogonal, hence

$$U^k \overset{?}{\le} -\,\theta^{-1}\left(\alpha-\beta\theta\right)\left\|\theta\mathbb{S}x^k-\mathbf{v}\right\|_M^2 + \left(\beta-\alpha^2\right)\|\mathbf{v}\|_M^2 . \tag{3}$$

To be precise, the Lemma 7.9 introduces the nearly orthogonality. Since from the almost sure convergence of the normalized iterate (Theorem 7.5), we can say that with probability 1 the sequence x^k satisfies the condition of Lemma 7.9. For such sequence x^k, and for an arbitrary $\delta \in (0, \pi/2)$, there exists a $N_{\delta,z}$ such that for all $k > N_{\delta,z}$,

$$\left\langle \mathbf{v}, \mathbb{S}x^k - \mathbb{S}z\right\rangle_M \le \sin\delta\,\|\mathbf{v}\|_M \left\|\mathbb{S}x^k-\mathbb{S}z\right\|_M .$$

From the inequality above, for all sufficiently large $k > N_{\delta,z}$,

$$U^k \le \theta^2\left(\beta-\alpha^2\right)\|\mathbb{S}z\|_M^2 - \theta\left(\alpha-\beta\theta\right)\left\|\mathbb{S}x^k-\mathbb{S}z\right\|_M^2 + 2\theta\tau_{\delta,z,k}\left\|\mathbb{S}x^k-\mathbb{S}z\right\|_M .$$

Here, $\tau_{\delta,z,k}$ is defined as

$$\tau_{\delta,z,k} = \left(\beta-\alpha^2\right)\left(\|\theta\mathbb{S}z-\mathbf{v}\|_M + \sin\delta\,\|\mathbf{v}\|_M\right) + \left(\alpha-\alpha^2\theta\right)\left\|\mathbb{S}z-\mathbb{S}z^k\right\|_M .$$

First step is to bound $\tau_{\delta,z,k}$ independent from k. The second term can be bounded using $\left\|\mathbb{S}z^k\right\|_M \le \|\mathbb{S}z\|_M$ and triangular inequality with $\theta^{-1}\mathbf{v}$ as the third point. After a suitable calculation, it is possible to define $\tilde{\tau}_{\delta,z} \leq \tau_{\delta,z,k}$ that satisfies

$$\theta \mathbb{S} z \to \mathbf{v}, \delta \to 0 \Rightarrow \tilde{\tau}_{\delta,z} \to 0$$

The second step is to make bound without the x^k term. We can bound U^k as

$$\begin{aligned} U^k &\leq \theta^2 \left(\beta - \alpha^2\right) \|\mathbb{S} z\|_M^2 - \theta \left(\alpha - \beta\theta\right) \left\|\mathbb{S} x^k - \mathbb{S} z\right\|_M^2 + 2\theta \tilde{\tau}_{\delta,z} \left\|\mathbb{S} x^k - \mathbb{S} z\right\|_M \\ &\leq \theta^2 \left(\beta - \alpha^2\right) \|\mathbb{S} z\|_M^2 + \frac{\theta}{\alpha - \beta\theta} \tilde{\tau}_{\delta,z}^2. \end{aligned}$$

However, note that this upper bound holds only at $k > N_{\delta,z}$ where $N_{\delta,z}$ also depends on the choice of the sequence x^k, thus such upper bound only works when the sequence $x^0, x^1, x^2, \ldots$ is fixed. To avoid this problem, take a limit supremum of U^k over k :

$$\limsup_{k\to\infty} U^k \leq \theta^2 \left(\beta - \alpha^2\right) \|\mathbb{S} z\|_M^2 + \frac{\theta}{\alpha - \beta\theta} \tilde{\tau}_{\delta,z}^2.$$

Furthermore, due to Cesàro mean,

$$\limsup_{k\to\infty} \left\{ \frac{1}{k} \left\|x^0 - z\right\|_M^2 + \frac{1}{k} \sum_{j=0}^{k-1} U^j \right\} \leq \theta^2 \left(\beta - \alpha^2\right) \|\mathbb{S} z\|_M^2 + \frac{\theta}{\alpha - \beta\theta} \tilde{\tau}_{\delta,z}^2.$$

Note that above equation holds with probability 1. By Fatou's lemma, we also have

$$\limsup_{k\to\infty} \mathbb{E} \left[\frac{1}{k} \left\|x^0 - z\right\|_M^2 + \frac{1}{k} \sum_{j=0}^{k-1} U^j \right] \leq \mathbb{E} \left[\limsup_{k\to\infty} \left\{ \frac{1}{k} \left\|x^0 - z\right\|_M^2 + \frac{1}{k} \sum_{j=0}^{k-1} U^j \right\} \right].$$

Thus, $\limsup_{k\to\infty} \mathbb{E}\left[k \left\|\frac{x^k}{k} - \frac{z^k}{k}\right\|_M^2\right]$ has a upper bound of

$$\limsup_{k\to\infty} \mathbb{E} \left[k \left\| \frac{x^k}{k} - \frac{z^k}{k} \right\|_M^2 \right] \leq \theta^2 \left(\beta - \alpha^2\right) \|\mathbb{S} z\|_M^2 + \frac{\theta}{\alpha - \beta\theta} \tilde{\tau}_{\delta,z}^2. \tag{4}$$

For the final step, choose z as the point $\theta \mathbb{S} z = \mathbf{v}$ to prove the statement (a), use minimum square norm property of variance to proce the statement (b). Then, take the limit of $\theta \mathbb{S} z \to \mathbf{v}, \delta \to 0$ to conclude the proof. □

Lemma 7.8. *Suppose θ-averaged operator $\mathbb{T} \colon \mathcal{H} \to \mathcal{H}$ has an infimal displacement vector $\mathbf{v}$. Consider a closed cone C_δ in $\mathcal{H}$ with $\delta \in (0, \pi/2)$, which is a set of vectors whose angle between them and $\mathbf{v}$ being less than $\pi/2 - \delta$.*

$$C_\delta = \left\{ x : \langle \mathbf{v}, x \rangle_M \geq \sin\delta \, \|\mathbf{v}\|_M \, \|x\|_M \right\}.$$

When the points $y, z \in \mathcal{H}$ satisfy that $\mathbb{S}y \in \mathbb{S}z + C_\delta$ and $\mathbb{S}y \neq \mathbb{S}z$, then the following inequality holds.

$$\langle -\mathbf{v}, y - z \rangle_M \le \cos\delta \, \|\mathbf{v}\|_M \, \|y - z\|_M .$$

Lemma 7.9. *Let $\mathbb{T}\colon \mathcal{H} \to \mathcal{H}$ be a θ-averaged operator with respect to $\|\cdot\|_M$. Let $\mathbf{v}$ be the infimal displacement vector of $\mathbb{T}$. Let $\mathbb{S} = \theta^{-1}(\mathbb{I} - \mathbb{T})$. Consider a sequence $y^0, y^1, y^2, \dots$ in $\mathcal{H}$ such that its normalized iterate converges strongly to $-\gamma\mathbf{v}$,*

$$\lim_{k\to\infty} \frac{y^k}{k} = -\gamma\mathbf{v},$$

for some $\gamma > 0$. Then, for any $\delta \in (0, \pi/2)$ and $z \in \mathcal{H}$, there exists $N_{\delta,z} \in \mathbb{N}$ such that, for all $k > N_{\delta,z}$,

$$\left\langle \mathbf{v}, \mathbb{S}y^k - \mathbb{S}z \right\rangle_M \le \|\mathbf{v}\|_M \left\|\mathbb{S}y^k - \mathbb{S}z\right\|_M \sin\delta.$$

Proof. Choose a point z in $\mathcal{H}$. To prove by contradiction, suppose that for any l, there exists $k_l > l$ such that

$$\mathbb{S}y^{k_l} \in \mathbb{S}z + C_\delta, \quad \mathbb{S}y^{k_l} \neq \mathbb{S}z.$$

The subsequence $y^{k_1}, y^{k_2}, y^{k_3}, \dots$ satisfies the inequality below for all l, due to Lemma 7.8.

$$\left\langle -\mathbf{v}, y^{k_l} - z \right\rangle_M \le \cos\delta \, \|\mathbf{v}\|_M \left\|y^{k_l} - z\right\|_M .$$

Divide each side by k_l and take a limit as $l \to \infty$. Since $\lim_{l\to\infty} y^{k_l}/k_l = -\gamma\mathbf{v}$ strongly,

$$\gamma \|\mathbf{v}\|_M^2 = \langle -\mathbf{v}, -\gamma\mathbf{v} \rangle_M \le \cos\delta \, \|\mathbf{v}\|_M \, \|-\gamma\mathbf{v}\|_M < \gamma \|\mathbf{v}\|_M^2 ,$$

which yields a contradiction.

Thus, when z is given, for any $\delta \in (0, \pi/2)$, there exist a $N_{\delta,z}$ such that for all $k > N_{\delta,z}$, it is either $\mathbb{S}y^k = \mathbb{S}z$ or $\mathbb{S}y^k \notin \mathbb{S}z + C_\delta$. As a conclusion, for all $k > N_{\delta,z}$,

$$\left\langle \mathbf{v}, \mathbb{S}y^k - \mathbb{S}z \right\rangle_M \le \sin\delta \, \|\mathbf{v}\|_M \left\|\mathbb{S}y^k - \mathbb{S}z\right\|_M .$$

□

7.4 Tightness of variance bounds

In this section, we provide examples for which the variance bound of Theorem 7.7 holds with equality and with a strict inequality. We then discuss how the geometry of range $(\mathbb{I} - \mathbb{T})$ influences the tightness of the inequality. Throughout this section, we consider the setting where the norm and inner product is $\|\cdot\|$-norm and $\langle \cdot, \cdot \rangle$, with $\mathcal{H} = \mathbb{R}^m$, $\mathcal{H}_i = \mathbb{R}$, and $\mathcal{I}$ follows uniform distribution on the set of standard unit vectors of $\mathcal{H}$. In this case, the smallest β we can choose is $\alpha = 1/m$.

7.4.1 Example: Theorem 7.7(b) holds with equality.

Consider the translation operator $\mathbb{T}(x) = x - \mathbf{v}$. When $x^0, x^1, x^2, \dots$ are the iterates of (RC-FPI) with $\mathbb{T}$, then

$$k\mathrm{Var}_M\left(\frac{x^k}{k}\right) = \alpha\,(1-\alpha)\,\|\mathbf{v}\|^2$$

for $k = 1, 2, \dots$, and the variance bound of Theorem 7.7 holds with equality.

7.4.2 Example: Theorem 7.7(b) holds with strict inequality.

Define $\mathbb{T}\colon \mathbb{R}^2 \to \mathbb{R}^2$ as

$$\mathbb{T}\colon\ (x, y) \mapsto \left(x - \frac{1+x-y}{2},\ y - \frac{1+y-x}{2}\right),$$

which is 1/2-averaged and has the infimal displacement vector $(1/2, 1/2)$.

When $(x^0, y^0), (x^1, y^1), (x^2, y^2), \dots$ are the iterates of (RC-FPI) with $\mathbb{T}$, then

$$\limsup_{k\to\infty}\ k\mathrm{Var}_M\left(\frac{(x^k, y^k)}{k}\right) = \frac{1}{24}. \tag{5}$$

On the other hand, the right hand side of the inequality in Theorem 7.7 (b) is

$$\alpha\,(1-\alpha)\,\|\mathbf{v}\|^2 = \frac{1}{2}\left(1 - \frac{1}{2}\right)\|\mathbf{v}\|^2 = \frac{1}{8}.$$

7.5 Relationship between the variance and the range set.

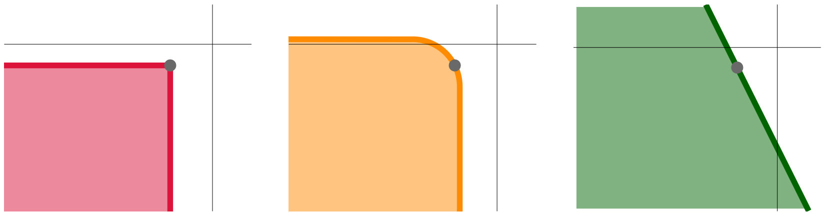
Consider the three convex sets A, B, and C in Figure 1 as a subset of $\mathcal{H} = \mathbb{R}^2$. The explicit definitions are

$$\begin{aligned} A &= \{(x, y) \mid x \le -10,\ y \le -5\} \\ B &= \left\{(x, y) \mid \mathrm{dist}\left((x, y), 3A\right) \le 2\sqrt{5^2 + 10^2}\right\} \\ C &= \{(x, y) \mid -2x - y \ge 25\}, \end{aligned}$$

where $\mathrm{dist}\left((x, y), 3A\right)$ denotes the (Euclidean) distance of (x, y) to the set $3A = \{(3x, 3y) \,|\, (x, y) \in A\}$. The minimum norm elements in each set are all identically equal to $(-10, -5)$.

Let $\mathbb{T} = \mathbb{I} - \theta\mathrm{Proj}$, where Proj denotes the projections onto A, B, and C. Then $\mathbb{T}$ is θ-averaged and $\mathrm{range}\,(\theta^{-1}(\mathbb{I} - \mathbb{T}))$ is equal to A, B, and C, respectively. These sets are designed for $\mathbb{T}$ to have the same infimal displacement vector. Figure 2 (left), shows that the normalized iterates of the three instances have different asymptotic variances despite identical $\mathbf{v}$. In the experiment, θ was set as 0.2, and as a consequence,

$\mathbf{v} = (-2, -1)$ is the infimal displacement vector for each experiments. (RC-FPI) is performed with $x^0 = (0, 0)$, $m = 2$ and $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{R}$.



(a) A (b) B (c) C

Fig. 1 Visualization A, B, and C as defined in Section 7.4. The grey dot is $\theta^{-1}\mathbf{v}$, where $\mathbf{v}$ is the infimal displacement vector of $\mathbb{T} = \mathbb{I} - \theta\mathrm{Proj}$.

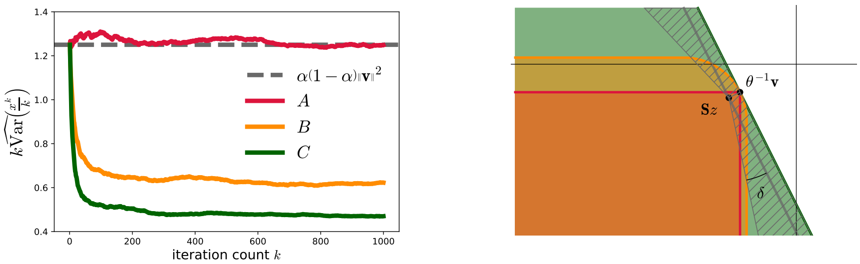


Fig. 2 (Left) Graph of $k\widehat{\mathrm{Var}}\left(x^k/k\right)$ by k, where $\widehat{\mathrm{Var}}\left(x^k/k\right)$ is the variance estimate with 10,000 samples. (Right) Visualization of A, B, and C as red, yellow, green regions and $D_{z,\delta}$ as the hatched area, where the sets are as defined in Section 7.4. We conjecture that the broader intersection with $D_{z,\delta}$ leads to smaller asymptotic variance.

We conjecture that the asymptotic variance is intimately related to the geometry of the set $\mathrm{range}\,(\theta^{-1}(\mathbb{I} - \mathbb{T}))$. For $z \in \mathbb{R}^n$ and $\delta > 0$, let

$$D_{z,\delta} = \left\{u \in \mathbb{R}^2 \middle| \langle \mathbf{v}, u - \mathbb{S}z\rangle \le \|\mathbf{v}\| \, \|u - \mathbb{S}z\| \sin\delta \right\}.$$

Lemma 7.9 states that eventually, $\mathbb{S}x^k \in D_{z,\delta}$ for sufficiently large k. Since $\mathbb{S}x^k \in \mathrm{range}\,(\theta^{-1}(\mathbb{I} - \mathbb{T}))$ for all k, the shaded region in the Figure 2 (right) depicting $D_{z,\delta} \cap \mathrm{range}\,(\theta^{-1}(\mathbb{I} - \mathbb{T}))$ actually shows the region where $\mathbb{S}x^k$ lies for large k. In the proof of Theorem 7.7, loosely speaking, we establish the upper bound using

$$-\theta\left(\alpha - \beta\theta\right)\left\|\mathbb{S}x^k - \theta^{-1}\mathbf{v}\right\|_M^2 \le 0.$$

Therefore, the variance can be strictly smaller than the upper bound when $\left\|\mathbb{S}x^k - \theta^{-1}\mathbf{v}\right\|_M^2$ is large, which can happen when the area of intersection $D_{z,\delta} \cap \mathrm{range}\,(\theta^{-1}(\mathbb{I} - \mathbb{T}))$ is large near $\theta^{-1}\mathbf{v}$. This can be observed in Figure 2, which shows that the range set having large intersection with $D_{z,\delta}$ have smaller asymptotic variance.

7.6 Infeasibility detection

In this section, we present the infeasibility detection method for (RC-FPI) using the hypothesis testing.

Theorem 7.10. *Let* $\mathbb{T}\colon \mathcal{H} \to \mathcal{H}$ *be* θ*-averaged with respect to* $\|\cdot\|_M$ *with* $\theta \in (0,1]$*. Let* $\mathbf{v}$ *be the infimal displacement vector of* $\mathbb{T}$*. Assume* $\mathcal{I}^0, \mathcal{I}^1, \ldots$ *is sampled IID from a distribution satisfying the uniform expected step-size condition* (1) *with* $\alpha \in (0,1]$*, and assume* (2) *holds with some* $\beta > 0$ *such that* $\beta < \alpha/\theta$*. Let* $x^0, x^1, x^2, \ldots$ *be the iterates of* (RC-FPI)*. Then*

$$\mathbb{P}\left(\left\|\frac{x^k}{k}\right\|_M \geq \varepsilon\right) \lesssim \frac{\left(\beta - \alpha^2\right)\delta^2}{k(\varepsilon - \alpha\delta)^2}$$

as $k \to \infty$*, where* $\mathbf{v}$ *is the infimal displacement vector of* $\mathbb{T}$*.*

Therefore, for any statistical significance level $p \in (0,1)$, the test

$$\left\|\frac{x^k}{k}\right\|_M \geq \varepsilon$$

with

$$k \gtrsim \frac{\left(\beta - \alpha^2\right)\delta^2}{p\left(\varepsilon - \alpha\delta\right)^2}$$

can reject the null hypothesis and conclude that $\|\mathbf{v}\|_M > \delta$, which implies that the problem is inconsistent.

For the proof of the Theorem 7.10, we begin with the simpler case where $\mathbf{v} \in \operatorname{range}\left(\mathbb{I} - \mathbb{T}\right)$. Let $\|\mathbf{v}\|_M \leq \delta$ be the null hypothesis with δ satisfying $\alpha\delta < \epsilon$. By the triangle inequality, Markov inequality, and Theorem 7.7, under the null hypothesis,

$$\begin{aligned}
\mathbb{P}\left(\left\|\frac{x^k}{k}\right\|_M \geq \varepsilon\right) &\leq \mathbb{P}\left(\left\|\frac{x^k}{k} + \alpha\mathbf{v}\right\|_M \geq \varepsilon - \alpha\delta\right) \\
&\leq \frac{1}{(\varepsilon - \alpha\delta)^2}\mathbb{E}\left[\left\|\frac{x^k}{k} + \alpha\mathbf{v}\right\|_M^2\right] \\
&\lesssim \frac{\left(\beta - \alpha^2\right)\delta^2}{k(\varepsilon - \alpha\delta)^2}
\end{aligned}$$

as $k \to \infty$.

When $\mathbf{v} \notin \operatorname{range}\left(\mathbb{I} - \mathbb{T}\right)$, we can still obtain the same (asymptotic) statistical significance with the same test and the same iteration count $k \gtrsim \frac{\left(\beta-\alpha^2\right)\delta^2}{p(\varepsilon-\alpha\delta)^2}$. The full proof of the general case is provided in my paper.