# Applications of Renormalization group method for generalized Kuramoto type models

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**Abstract**

The Kuramoto type models such as the Lohe sphere model describes the mechanics of the particles that are coupled to each other. These models mathematically describes phenomenons such as synchronization or clustering. However, due to its complexity, its exact solution and converging conditions remains to be hidden. In this paper, we applied the Renormalization Group method to analysis the system. Under non-resonance conditions, we have calculated up to the second order RG system, which gives more accurate numerical solutions. From the symmetry between stable manifolds of original system and renormalized system, it was possible to find the state that the Lohe sphere model converges under clustering conditions.

## Introduction

In complex systems where objects interact, we can easily observe natural phenomena where the periodical phase speed of objects in a system becoming the same. Such phenomenon is called 'synchronization phenomenon'. The Kuramoto type models describes the synchronization mathematically. However, despite the importance of the Kuramoto type models, their exact solutions and convergence analysis are not yet discovered due to its complexity.

The convergence analysis by computing the numerical solutions generates a secular term in the approximate solution. The secular term diverges as the time goes to infinity, while the exact solution is bounded. Renormalization Group method cancels out the secular term using the envelope theory. With Renormalization Group method, we were able to find the state of the system if it converges.

## Main Objectives

1. Integration results on forms involving matrix exponential terms.
2. RG equations of the Lohe sphere model with the non-resonance conditions.
3. RG equations of the Lohe sphere model with the clustering conditions.
4. Computation of the stable manifold of the Lohe sphere model with the clustering conditions.

## Preliminaries

### The Kuramoto type models

The Lohe sphere model is a model that describes the dynamics of particles on n-th dimensional unit sphere, with the coupling applied. The 2-dimensional case responds to the Kuramoto model, which describes the collective dynamics on the unit circle. Let $\theta_i \in \mathbb{S}$ be the angular position of the $i$-th particle on unit circle. The *Kuramoto model* and the dynamics of $\theta_i$ reads as the following :

$$\dot{\theta}_i = \omega_i + \frac{\kappa}{N}\sum_{k=1}^{N}\sin\left(\theta_j - \theta_i\right), \quad i = 1, \cdots, N, \tag{1}$$

where $\omega_i \in \mathbb{R}$ is the natural frequency of the $i$-th particle, and $\kappa$ is the coupling strength [4]. Now, generalize the Kuramoto model to the $d$-dimension. Let $x_i \in \mathbb{S}^d$ be the position of the $i$-th Lohe particle. The *Lohe sphere model* and the dynamics of $x_i$ reads as the following :

$$\dot{x}_i = \Omega_i x_i + \frac{\kappa}{N}\sum_{k=1}^{N}\left(x_k - \frac{\langle x_i, x_k\rangle}{\langle x_i, x_i\rangle}x_i\right), \quad i = 1, \cdots, N, \tag{2}$$

where $\Omega_i$ is the natural frequency matrix of the $i$-th particle which is skew-symmetric, and $\kappa$ is the coupling strength [5].

### The Renormalization Group method

The key idea of the Renormalization Group method is approximating the exact solution at every time and position, to generate the approximated flow. The differential equation generated by approximated flow , which is called as RG equation, gives the approximated solution without any secular term[2]. Reducing the secular term is useful for long-time analysis, including the convergence analysis.

Consider ODE

$$\dot{x} = Fx + \epsilon g(t, x, \epsilon) \tag{3}$$

where $\epsilon \in \mathbb{R}$ is a small parameter, $F$ is a $n \times n$ matrix whose all eigenvalues lie on the imaginary axis or the left half plane, and $g(t, x, \epsilon)$ is $C^\infty$ class vector field with each term in its power expansion $g(t, x, \epsilon) = g_1(t, x) + \epsilon g_2(t, x) + \cdots$ are periodic in $t$ and polynomial in $x$.

**Definition 1.** *We define the* $m$th order RG equation *for* (3) *as*

$$\frac{dA}{dt} = \dot{A} = \epsilon R_1(A) + \epsilon^2 R_2(A) + \cdots + \epsilon^m R_m(A), \quad A \in \mathbb{R}^n, \tag{4}$$

*where $X(t) = e^{Ft}A$ is a solution of the unperturbed part $\dot{x} = Fx$, $G_i$ is an $i$-th order part of perturbation term $g$, and*

$$R_1(A) := \lim_{t\to\infty}\frac{1}{t}\int^t X(s)^{-1}G_1(s, X(s)A)ds.$$

**Theorem 1** (Existence of invariant manifolds)**.** *Let $\epsilon^k R_k(A)$ be a first non-zero term in the RG equation. If the vector field $\epsilon^k R_k(A)$ has a normally hyperbolic invariant manifold $N$, then the original equation also has a normally hyperbolic invariant manifold $N_\epsilon$, which is diffeomorphic to N, for sufficiently small $|\epsilon|$. In particular, the stability of $N_\epsilon$ coincides with that of $N$. [1]*

## Results

First, key result on integration on matrix exponential is the following theorem.

**Theorem 2.** *When $A, B$ are non-singular skew-symmetric and $a, b \in \mathbb{R}^d$,*

$$\lim_{t\to\infty}\frac{1}{t}\int^t e^{-As}ab^T e^{Bs}ds = \sum_{\lambda:\det(C-\lambda I)=0} P_\lambda^A a\left(P_\lambda^B b\right)^\dagger, \tag{5}$$

*where $P_\lambda^A$ represents a projection on eigenspace of $A$ with eigenvalue $\lambda$.*

Now, define the non-resonance conditions on the Lohe sphere model.

**Definition 2.** *The non-resonance conditions are defined as :*

*1. $\Omega_i\Omega_j$ is symmetric for any $i, j$.*

*2. $\Omega_i - \Omega_j$ is nonsingular whenever $i \neq j$.*

*3. $\Omega_i - \Omega_j$ and $\Omega_j - \Omega_k$ do not share any eigenvalues whenever $i, j, k$ are distinct indices.*

From the theorem 2, we were able to compute $R_1$ and $R_2$ of the Lohe sphere model with the non-resonance conditions.

$$\begin{aligned} R_1(y)|_i &= y_i\left(1 - y_i^T y_i\right) \\ R_2(y)|_i &= \sum_{k\neq i}^{N}\left(1 + ||y_k||^2\right)\left(\Omega_i - \Omega_k\right)^{-1} y_i \end{aligned} \tag{6}$$

Next, we give the Lohe sphere model with the clustering condition.

$$\dot{x_{ij}} = \Omega_i x_{ij} + \frac{\kappa}{N}\left(I - x_{ij}x_{ij}^T\right)\sum_{k=1}^{m}\sum_{l=1}^{n_k} x_{kl}, \qquad N = \sum_{k=1}^{m} n_k. \tag{7}$$

The clustering condition represents the situation when there are small number of natural frequency choices. In the [3], it is previously shown that every particles converges to a single particle if their natural frequencies are identical. The clustering condition generalizes to the $m$ choices of natural frequencies instead of one. The 1st order RG equation of the Lohe sphere model with the clustering condition is,

$$\dot{y_{ij}} = \epsilon R_1(y)|_{ij} = \frac{\kappa}{N}\left(I - y_{ij}y_{ij}^T\right)\sum_{l=1}^{n_i} y_{il}. \tag{8}$$

In 2-dimensional Kuramoto case, we may rewrite the (8) with polar coordinates,

$$\dot{\phi_{ij}} = \sum_{l=1}^{n}\sin(\phi_{il} - \phi_{ij}). \tag{9}$$

Consider the manifold satisfying $\phi_{ij} = \theta_i$ for every $j$, for each $i$, which corresponds to the state where particles with same natural frequencies cluster to one. The Jacobian of (9) on this manifold is $\mathbf{1}\mathbf{1}^T - nI$ for each $i$, which contains $n-1$ negative eigenvalues and one zero eigenvalue. Thus, it is a stable manifold. By the symmetry of the stable manifolds 1, $\phi_{ij} = \theta_i$ for every $j$ also gives the stable manifold for the original problem.

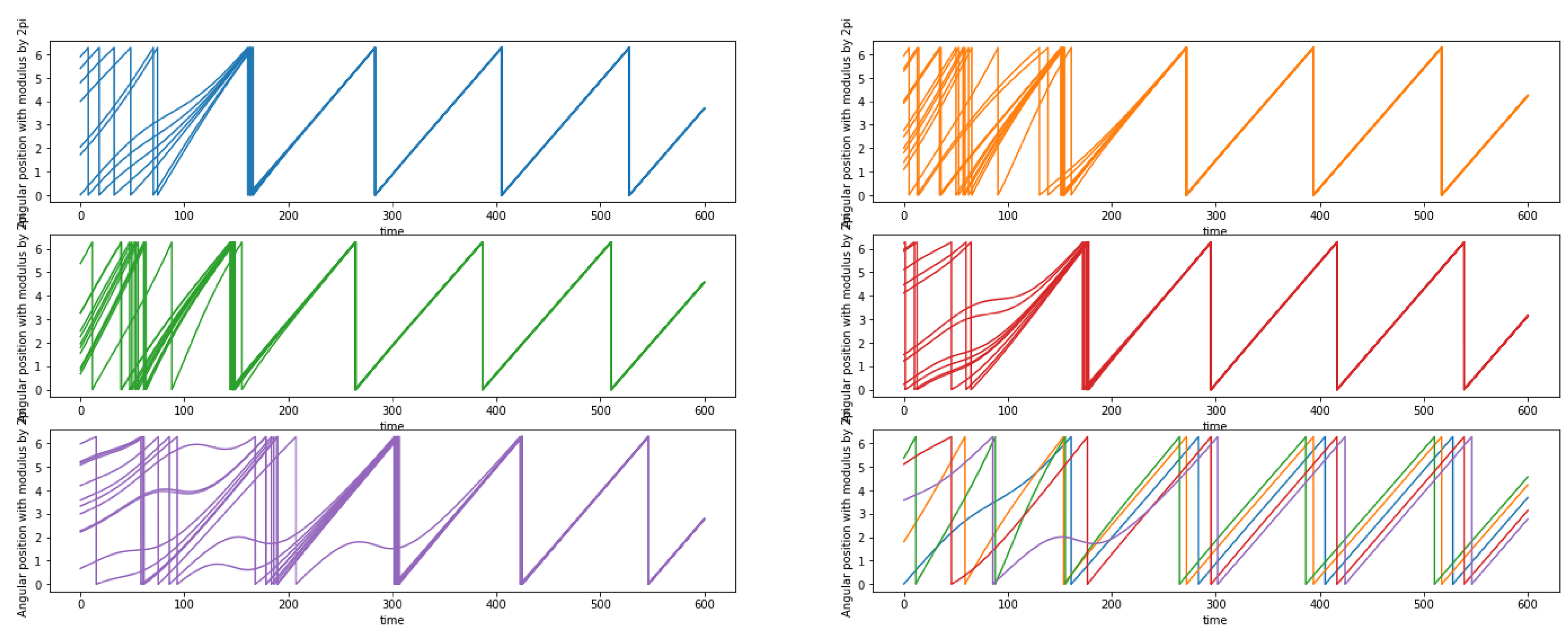

**Figure 1:** Result of the simulation. Group of $m = 5$ were used, with the coupling strength as $\kappa = 0.05$. The natural frequencies are set as $(n_i, \omega_i)$ =(7, 0.05), (12, 0.07),(11, 0.08),(9, 0.03),(11, 0.02).

Figure shows the results of the simulation of the 2-dimensional Lohe sphere model with the clustering condition. From the first five graphs, we could observe that particles in the same group converges and acts like a single particle over time. This clustering phenomenon which coincides with the stable manifold we computed. For the last graph we could check the synchronization phenomenon but particles in distinct groups do not cluster as one.

## Conclusions

- When the non-resonance conditions are given, we have computed through second order RG equation. The second order RG equation was sufficient to use the symmetry of stable manifold.
- With the clustering conditions, we were able to find the stable manifolds using the first order RG equation. The state where the particles with equal natural frequencies being clustered as one induces the stable manifold. Additionally, we could observe that the system actually converges to this stable manifold.

## Forthcoming Research

For further analysis on the clustering condition, we plan to study on the symmetry of convergence radius of the stable manifolds. It will bring the convergence conditions on the Lohe sphere model by using convergence radius of the stable manifolds of RG equations.

## References

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