APPLICATIONS OF RENORMALIZATION GROUP METHOD FOR GENERALIZED KURAMOTO TYPE MODELS

JINHEE PAENG, KIBEOM MYOUNG, SEUNGMOK LEE, SEUNG YEAL HA, AND JAEYOUNG YOON

ABSTRACT. The Kuramoto type models such as the Lohe sphere model describes the mechanics of the particles that are coupled to each other. These models mathematically describes phenomenons such as synchronization or clustering. However, due to its complexity, its exact solution and converging conditions remains to be hidden. In this paper, we applied the Renormalization Group method to analysis the system. Using the symmetry between invariant manifolds of Renormalized Group equation and original system, it was possible to show that the manifold corresponding to the clustering or complete synchronization among particles is indeed stable.

1. INTRODUCTION

In complex systems where objects interact, we can easily observe natural phenomena around us that show a more consistent and uniform pattern of motion of the objects over time. Such phenomenon is called a 'clustering phenomenon', examples of which include a flocking phenomenon and a synchronization phenomenon. Flocking phenomenon refers to the motion of the objects at the same speed in a disordered behavior, and synchronization phenomenon refers to the phase and phase speed of objects in a system with periodicity becoming the same. The study of clustering phenomena in such complex systems has been positioned as a very important factor in understanding fields such as biology [6], statistical physics, and sociology.

Flocking research, basically based on mathematical modeling, was first attempted in 1995 by Hungarian statistical physicist Vicsek and his co-researchers [10]. Soon after that, in 2007, Cucker and Smale proposed a new model, which is an enhancement of a previous Vicsek model. Then it followed by various models for explaining synchronization which have also been established by many mathematicians. The two models to be mentioned in this study are the Kuramoto model and the Lohe model.

The Kuramoto model was first proposed by Yoshiki Kuramoto in 1975 [8]. This model is abstracted assuming a biochemical situation where numerous coupled oscillators interact, and the coupling strength between the objects is described as a governing equation. The simplest form of the Kuramoto model limited to the one-dimensional phase space contains

Date: June 26, 2023.

2020 *Mathematics Subject Classification.* 39A30, 34E10, 65L05.

Key words and phrases. Kuramoto model, Lohe sphere model, Renormalization Group method, Stable manifold.

Acknowledgment. The work of J. Paeng was supported by Faculty of Liberal Education Seoul National University (A2021-0046).

the content that it is coupled by the following sinusoidal function.

$$\dot{\theta}_i = \omega_i + \frac{\kappa}{N}\sum_{k=1}^{N}\sin(\theta_j - \theta_i), \quad i = 1, \cdots, N$$

The Lohe model deals with the interaction of particles on a unit sphere. The relationship between particles can be extended similarly to the Kuramoto model. The interaction relation between particles in the Lohe model is as follows.

$$\dot{x_i} = \Omega_i x_i + \frac{\kappa}{N}\sum_{k=1}^{N}\left(x_k - \frac{\langle x_i, x_k\rangle}{\langle x_i, x_i\rangle}x_i\right), \quad i = 1, \cdots, N$$

In the above two models, regardless of the phase and phase velocity of the initial particles, a phenomenon in which the phase velocity becomes the same over time or a clustering phenomenon in the same phase is observed. In the sense that such a surprising result is found in spite of a very simple analytical structure, it can represent numerous natural phenomena as well as being actively used in related fields. For example, the Kuramoto model is widely used as a synchronization model in statistical physics.

Despite the importance of the Lohe sphere model or the Kuramoto model, their exact solutions are not yet discovered and even seems to be impossible. Additionally, the converging condition remains to be open in general case. There were some numerical approaches to solve the convergence of the system, but previous numerical approaches works only when the coupling is strong.

The convergence analysis by computing the numerical solutions are difficult due to the diverging error term. The most common method, a perturbation method, generates a secular term in the approximate solution. The secular term diverges as the time goes to infinity, while the exact solution is bounded. This is the main difficulty in numerical analysis, and the reason why the previous approaches works when the coupling is strong.

In this paper, we will apply Renormalization Group method. Renormalization Group method cancels out the secular term using the envelope theory. The approximation of the exact solution is computed at every time and position, generating the approximated flow. The differential equation generated by approximated flow , which is called as RG equation, gives the approximated solution without any secular term [3]. Reducing the secular term is useful for long-time analysis, including the convergence analysis.

By application of RG method, we found the stable manifold that corresponds to the clustering phenomenon of the Lohe sphere model. This result yields from the symmetry of the stable manifold of RG equation and original system. Since the clustering phenomenon corresponds to the stable manifold, we assume that under some conditions the system will converge and show the clustering formation over time. We will verify this by simulation later in the paper.

The rest of this paper is organized as follows. In section 2, we will review on previous works, including definitions of the Kuramoto type models and the Renormalization Group method. In section 3, we will present lemmas on integration involving matrix exponentials, which are used later in this paper. In section 4, we will focus on the clustering phenomenons among particles with identical natural frequencies using the symmetry of the stable manifolds. In section 5, we present a simulation result on simple example, which coincides with the theoretical result of section 4. In section 6, RG equations and approximated solution of the Lohe sphere model with the non-resonance conditions will be computed.

2. Preliminaries

In this section, we review on the Kuramoto type models. The *Lohe sphere model* is a model that describes the dynamics of particles on n-th dimensional unit sphere, with the coupling applied. The 2-dimensional case responds to the *Kuramoto model*, which describes the collective dynamics on the unit circle. After the reviews on the Kuramoto type models, we will cover about the Renormalization Group method.

2.1. **The Kuramoto type models.**

2.1.1. *The Kuramoto model.* Let $\theta_i \in \mathbb{S}$ be the angular position of the i-th particle on unit circle. The *Kuramoto model* and the dynamics of θ_i reads as follows:

$$\dot{\theta}_i = \omega_i + \frac{\kappa}{N}\sum_{k=1}^{N} \sin(\theta_j - \theta_i), \quad i = 1, \cdots, N, \tag{2.1}$$

where $\omega_i \in \mathbb{R}$ is the natural frequency of the i-th particle, and κ is the coupling strength [7].

2.1.2. *The Lohe sphere model.* Let $x_i \in \mathbb{S}^d$ be the position of the i-th Lohe particle. The *Lohe sphere model* and the dynamics of x_i reads as follows:

$$\dot{x_i} = \Omega_i x_i + \frac{\kappa}{N}\sum_{k=1}^{N}\left(x_k - \frac{\langle x_i, x_k\rangle}{\langle x_i, x_i\rangle} x_i\right), \quad i = 1, \cdots, N, \tag{2.2}$$

where Ω_i is the natural frequency matrix of the i-th particle which is skew-symmetric, and κ is the coupling strength [9]. Next, we present a conserved quantity of the Lohe sphere model. In Lohe sphere model, each particle's distance from the origin stays at constant 1.

Lemma 2.1. *Let x_i be a solution to* (2.2) *with initial point $x_i(0)$ with $\|x_i(0)\| = 1$. Then, $\|x_i(t)\| = 1$ for any positive t.*

Proof. From each Ω_j being skew-symmetric, $\langle x_i, \Omega_i x_i\rangle = 0$ since

$$\langle x_i, \Omega_i x_i\rangle = \left\langle {\Omega_i}^T x_i, x_i\right\rangle = -\langle \Omega_i x_i, x_i\rangle = -\langle x_i, \Omega_i x_i\rangle.$$

Now, differentiate the square of the norm $\|x_i(t)\|^2$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|x_i(t)\|^2 &= \left\langle x_i(t), \dot{x_i(t)}\right\rangle \\ &= \langle x_i(t), \Omega_i x_i(t)\rangle + \frac{\kappa}{N}\sum_{k=1}^{N}\left(\langle x_i(t), x_k(t)\rangle - \langle x_i(t), x_k(t)\rangle\right) = 0.\end{aligned}$$

Thus, norm of x_i stays constant, concluding $\|x_i(t)\| = 1$ for any positive time t. □

In this paper, we will only consider the case where each particle starts at the unit sphere. From the above lemma, each particle stays at the unit sphere. Hence, the system (2.2) can be rewritten as the following:

$$
\begin{aligned}
\dot{x_i} &= \Omega_i x_i + \frac{\kappa}{N}\sum_{k=1}^{N}\left(x_k - \langle x_i, x_k\rangle x_i\right) \\
&= \Omega_i x_i + \frac{\kappa}{N}\sum_{k=1}^{N}\left(x_k - x_i x_i^T x_k\right) \\
&= \Omega_i x_i + \frac{\kappa}{N}\left(I - x_i x_i^T\right)\sum_{k=1}^{N} x_k,
\end{aligned}
\tag{2.3}
$$

where I is $(d+1)\times(d+1)$ identity matrix.

2.2. **The Renormalization Group method.** In this subsection, we introduce Renormalization Group (RG) method and some key theorems [3].

Let F be an $n \times n$ matrix whose all eigenvalues lie on the imaginary axis or the left half plane. We assume that at least one eigenvalue is on the imaginary axis. Let $g(t, x, \epsilon)$ be a time-dependent vector field on $\mathbb{R}^n$ which is of C^∞ class with respect to t, x and ϵ. Let $g(t, x, \epsilon)$ has a formal power series expansion in ϵ, $g(t, x, \epsilon) = g_1(t, x) + \epsilon g_2(t, x) + \cdots$. We assume that $g_i(t, x)$'s are periodic in t and polynomial in x. Now consider an ODE

$$
\begin{aligned}
\dot{x} &= Fx + \epsilon g(t, x, \epsilon) \\
&= Fx + \epsilon g_1(t, x) + \epsilon^2 g_2(t, x) + \cdots ,
\end{aligned}
\tag{2.4}
$$

where $\epsilon \in \mathbb{R}$ is a small parameter. Substitute x by $x = x^{(0)} + \epsilon x^{(1)} + \epsilon^2 x^{(2)} + \cdots$ and expand the right-hand side of the above equation with respect to ϵ, we get a series of ODEs of $x^{(0)}, x^{(1)}, x^{(2)}, \ldots$:

$$
\begin{aligned}
\dot{x^{(0)}} &= Fx^{(0)}, \\
\dot{x^{(1)}} &= Fx^{(1)} + G_1(t, x^{(0)}), \\
&\vdots \\
\dot{x^{(i)}} &= Fx^{(i)} + G_i(t, x^{(0)}, x^{(1)}, \cdots, x^{(i-1)}),
\end{aligned}
$$

where the inhomogeneous terms G_i's are smooth fucntion of $t, x^{(0)}, \cdots, x^{(i-1)}$. $G_1, G_2, \cdots$, which are given by

$$
\begin{aligned}
G_1(t, x^{(0)}) &= g_1(t, x^{(0)}), \\
G_2(t, x^{(0)}, x^{(1)}) &= \frac{\partial g_1}{\partial x}(t, x^{(0)})x^{(1)} + g_2(t, x^{(0)}), \\
&\vdots
\end{aligned}
$$

There is a solution of the unperturbed part $\dot{x^{(0)}} = Fx^{(0)}$ by $x^{(0)}(t) = X(t)y$, where $X(t) = e^{Ft}$ and $y \in \mathbb{R}^n$ is an initial value. With this $x^{(0)}$, the equation of $x^{(1)}$ is rewritten as

$$
\dot{x^{(1)}} = Fx^{(1)} + G_1(t, X(t)y).
$$

The solution of above equation is

$$x^{(1)} = X(t)X(\tau)^{-1}h + X(t)\int_{\tau}^{t} X(s)^{-1}G_1(s, X(s)y)ds,$$

where $h \in \mathbb{R}^n$ is an initial value at an initial time $\tau \in \mathbb{R}$. Define $R_1(y)$ and $h_t^{(1)}(y)$ by

$$R_1(y) := \lim_{t\to\infty} \frac{1}{t}\int^{t} X(s)^{-1}G_1(s, X(s)y)ds,$$
$$h_t^{(1)}(y) := X(t)\int^{t} (X(s)^{-1}G_1(s, X(s)y) - R_1(y))ds,$$

respectively. We can verify that $R_1(y)$ is well defined. With these, when we replace h with $h_\tau^{(1)}(y)$, above equation is rewritten as

$$x^{(1)} := x_1(t, \tau, A) = h_t^{(1)}(y) + X(t)R_1(y)(t - \tau).$$

In this, one part $h_t^{(1)}(y)$ is bounded uniformly in $t \in \mathbb{R}$ (see [1]), and the other part $X(t)R_1(y)(t - \tau)$ is linearly increasing in t. We call this as the *secular term*. We can find solution of $x^{(2)}$ similarly with

$$R_2(y) := \lim_{t\to\infty} \frac{1}{t}\int^{t} \left[X(s)^{-1}G_2(s, X(s)y, h_s^{(1)}(y)) - X(s)^{-1}(Dh_s^{(1)})_y R_1(y)\right] ds,$$
$$h_t^{(2)}(y) := X(t)\int^{t} \left[X(s)^{-1}G_2(s, X(s)y, h_s^{(1)}(y)) - X(s)^{-1}(Dh_s^{(1)})_y R_1(y) - R_2(y)\right] ds.$$

Note that $x^{(3)}, x^{(4)}, \cdots$ can also be calculated in similar steps, but since we only use up to 2nd order in this paper, we omit the higher order results. Now the curve $x^{(0)} + \epsilon x^{(1)} + \epsilon^2 x^{(2)}$ is an approximated solution for the original system. Each approximated solutions are parameterized with the initial time and starting point τ, y. We aim to choose the starting point $y = y(\tau)$ so that the curve $x^{(0)} + \epsilon x^{(1)} + \epsilon^2 x^{(2)}$ is independent of τ. It can be written as

$$\frac{d}{d\tau}|_{\tau=t}(x^{(0)} + \epsilon x^{(1)}(t, \tau, y(\tau)) + \epsilon^2 x^{(2)}(t, \tau, y(\tau))) = 0.$$

This equation is called the *RG condition*, and it gives an ODE as follows [3] :

Definition 2.1. *We define the* mth order RG equation *for* (2.4) *as*

$$\frac{dy}{dt} = \dot{y} = \epsilon R_1(y) + \epsilon^2 R_2(y) + \cdots + \epsilon^m R_m(y), \quad y \in \mathbb{R}^n. \tag{2.5}$$

We define the mth order RG transformation $\alpha_t : R^n \to R^n$ *as*

$$\alpha_t(y) = X(t)y + \epsilon h_t^{(1)}(y) + \cdots + \epsilon^m h_t^{(m)}(y). \tag{2.6}$$

Furthermore, when $y = y(t, t_0, \xi)$ is a solution of mth order RG equation with initial time t_0 and initial value ξ, the curve $\tilde{x}(t) = \tilde{x}(t, t_0, \xi)$ defined by

$$\tilde{x} = \alpha_t(y(t, t_0, \xi)) = X(t)y(t, t_0, \xi) + \epsilon h_t^{(1)}(y(t, t_0, \xi)) + \cdots + \epsilon^m h_t^{(m)}(y(t, t_0, \xi)). \tag{2.7}$$

is an approximate solution for the original system with C^1 approximation.

Theorem 2.1 (Error Estimate [1])**.** *There exist positive constants ϵ_0, C, T and a compact subset $V = V(\epsilon) \subset \mathbb{R}^n$ with the origin included such that for $\forall|\epsilon| < \epsilon_0$, every solution $x(t)$ of (2.4) and its mth order RG approximated solution $\tilde{x}(t)$ with $x(0) = \tilde{x}(0) \in V(\epsilon)$ satisfy*

$$\|x(t) - \tilde{x}(t)\| < C\epsilon^m, \quad 0 \leq t \leq T/\epsilon.$$

Also, it is possible to determine stability of an invariant manifold of a system using the stability of an invariant manifold of non-zero RG equation.

Theorem 2.2 (Existence of invariant manifolds [2])**.** *Let $\epsilon^k R_k(A)$ be a first non-zero term in the RG equation. If the vector field $\epsilon^k R_k(A)$ has a normally hyperbolic invariant manifold N, then the original equation also has a normally hyperbolic invariant manifold N_ϵ, which is diffeomorphic to N, for sufficiently small $|\epsilon|$. In particular, the stability of N_ϵ coincides with that of N.*

3. Integrations involving matrices

In this section, we will cover about integration involving exponential of matrices of form

$$\int^t e^{-As} V e^{Bs} ds$$

with matrices A, B being skew-symmetric and non-singular, and V being matrix of rank 1. We define vectors a, b as a pair of vectors that satisfies $V = ab^T$. Let's build a block matrix $C = \begin{pmatrix} A & V \\ O & B \end{pmatrix}$ with O as a zero matrix. Then, an exponential of Ct is :

$$e^{Ct} = \begin{pmatrix} e^{At} & F(t) \\ O & e^{Bt} \end{pmatrix} \tag{3.8}$$

with some function $F(t)$. Use the relation $\frac{d}{dt}e^{Ct} = Ce^{Ct}$, we can derive the following equation.

$$\frac{d}{dt}F(t) = AF(t) + Ve^{Bt}$$

By solving this equation, $F(t)$ is

$$F(t) = \int^t e^{A(t-s)} V e^{Bs} ds. \tag{3.9}$$

Thus, it is possible to calculate $\int^t e^{-As} V e^{Bs} ds$ by calculating e^{Ct}. We perform this calculation with three steps. First we compute a Jordan form of the matrix C, then modify in the block matrix shape similar to the Jordan form, and lastly compute the desired function.

3.1. **Jordan form of the matrix.**

First, we will find the Jordan form of matrix C for exponential computation. Since the characteristic polynomial of C is identical to the characteristic polynomial of $E = \begin{pmatrix} A & O \\ O & B \end{pmatrix}$, the diagonal components of the Jordan form of C are eigenvalues of A and B according to its algebraic multiplicity.

Now, consider λ, which is one of eigenvalues of either A or B. To calculate Jordan block $J(\lambda)$, we need to compare geometric and algebraic multiplicity.

(1) Only A has λ as an eigenvalue.
In this case, Jordan blocks of λ has size 1. With A being diagonalizable, λ has same algebraic and geometric multiplicity of A. Say λ and multiplicity m of A. Since B does not has λ as an eigenvalue, algebraic multiplicity of λ of C is also m. Consider set of independent vectors

$$\left\{ \begin{pmatrix} u_\lambda^i \\ 0 \end{pmatrix} : u_\lambda^i \text{ is an eigenvector of } A \text{ with an eigenvalue } \lambda, i = 1, ..., m \right\}$$

which is a set of m eigenvectors of C with λ eigenvalue. This set forms eigenspace of λ of C with dimension m. Thus, λ has same algebraic and geometric multiplicity of C, making Jordan part of λ as $J_1(\lambda) \bigoplus J_1(\lambda) \bigoplus ...J_1(\lambda)$.

(2) Only B has λ as an eigenvalue.
This case is almost identical to the first case, with set of independent eigenvectors of λ of C as :

$$\left\{ \begin{pmatrix} -(A - \lambda I)^{-1} V v_\lambda^i \\ v_\lambda^i \end{pmatrix} : v_\lambda^i \text{ is an eigenvector of } B \text{ with an eigenvalue } \lambda, i = 1, ..., n \right\}$$

Since $A - \lambda I$ is invertible, this set is well defined, and forms eigenspace of λ of C with dimension n where n is algebraic multiplicity of λ of B and C. Thus, Jordan part of λ is $J_1(\lambda) \bigoplus J_1(\lambda) \bigoplus ...J_1(\lambda)$.

(3) Both A and B have λ as an eigenvalue, When $V = ab^T$,
   (a) a is in column space of $A - \lambda I$ or b is in row space of $B - \lambda I$.
   In this case, say row echelon form of $A - \lambda I$ as $R_A = E_A(A - \lambda I)$, and $B - \lambda I$ as $R_B = E_B(B - \lambda I)$. then,

$$\begin{pmatrix} E_A & O \\ O & E_B \end{pmatrix} (C - \lambda I) = \begin{pmatrix} R_A & E_A ab^T \\ O & R_B \end{pmatrix}.$$

   If a is in column space of $A - \lambda I$, $E_A ab^T$ can have nonzero row only when R_A has nonzero row. If b is in row space of $B - \lambda I$, then each row of $E_A ab^T$ is a linear combination of rows in R_B. In either case, we can conclude that nullity of $C - \lambda I$ is identical to the sum of nullity of $A - \lambda I$ and $B - \lambda I$, which is also algebraic multiplicity of λ. Thus, Jordan part of λ is $J_1(\lambda) \bigoplus J_1(\lambda) \bigoplus ...J_1(\lambda)$. The corresponding vectors are the following :

$$\left\{ \begin{pmatrix} u_\lambda^i \\ 0 \end{pmatrix} : i = 1, ..., m \right\} \cup \left\{ \begin{pmatrix} -(b^T v_\lambda^i) p_\lambda \\ v_\lambda^i \end{pmatrix} : i = 1, ..., n \right\},$$

   where p_λ is some vector such that satisfies $(A - \lambda I)p_\lambda = a$ when a is in column space of $A - \lambda I$, or

$$\left\{ \begin{pmatrix} u_\lambda^i \\ 0 \end{pmatrix} : i = 1, ..., m \right\} \cup \left\{ \begin{pmatrix} 0 \\ v_\lambda^i \end{pmatrix} : i = 1, ..., n \right\},$$

   when b is in row space of $B - \lambda I$.
   (b) Otherwise
   In other case, there exists nonzero row of $E_A ab^T$ with R_A having zero row exists. Such rows are linearly independent from rows of R_B, while each being equal up to constant factor. This leads to the result of that the nullity of $C - \lambda I$ is smaller by 1 to the sum of nullity of $A - \lambda I$ and $B - \lambda I$. Thus, Jordan

part of λ is $J_2(\lambda)\bigoplus J_1(\lambda)\bigoplus ...J_1(\lambda)$, with only one block being size of 2. The corresponding vectors are the following :

$$\left\{\begin{pmatrix} u_\lambda^i \\ 0 \end{pmatrix} : i = 1, ..., m\right\} \cup \left\{\begin{pmatrix} (b^T v_\lambda^1) q_\lambda \\ v_\lambda^1 \end{pmatrix}\right\} \cup \left\{\begin{pmatrix} 0 \\ (b^T v_\lambda^1) v_\lambda^i - (b^T v_\lambda^i) v_\lambda^1 \end{pmatrix} : i = 2, ..., n\right\},$$

where q_λ is some vector such that satisfies $(A-\lambda I)^2 q_\lambda = -(A-\lambda I)a$, and $u_\lambda^{(1)}$ is chosen from eigenspace as $(A-\lambda I)q_\lambda + a$.

3.2. **Modification of Jordan form.** From the results from Jordan form, we can separate C into diagonalizable part and nilpotent part as diagonal and superdiagonal parts. Let such separation as $C = M + N$, where M is diagonalizable part and N is nilpotent part. From the Jordan form, we can easily show that M and N commutes and $N^2 = O$.

Before further calculation, by applying permutation and change of basis on eigenspaces, we may write $C = SJS^{-1}$ as following :

$$S = \begin{pmatrix} U_A & R \\ O & U_B \end{pmatrix}, S^{-1} = \begin{pmatrix} U_A^{-1} & -U_A^{-1} R U_B^{-1} \\ O & U_B^{-1} \end{pmatrix}, J = \begin{pmatrix} \Lambda_A & H \\ O & \Lambda_B \end{pmatrix}, \tag{3.10}$$

where $U_A = [u_1|u_2|...|u_d]$ and $U_B = [v_1|v_2|...|v_d]$ are unitary matrix formed with their orthonormal eigenvectors as its column vectors, and R and H follows the following rule.

For each eigenvector v_j of B with its eigenvalue if λ, the column vector R_j, H_j of R, H is

(1) If A does not contain λ as an eigenvalue, then $H_j = 0$ and $R_j = -(A-\lambda I)^{-1} V v_j$.
(2) When A contains an λ as an eigenvalue and a is in column space of $A - \lambda I$, then $H_j = 0$ and $R_j = -(b^T v_j) p_\lambda$.
(3) When A contains an λ as an eigenvalue and b is in row space of $B - \lambda I$, then $H_j = 0$ and $R_j = 0$.
(4) Otherwise, $R_j = (b^T v_j) q_\lambda$ and $H_j = U_A^{-1}\left((A-\lambda I)q_\lambda + a\right)(b^T v_j)$.

3.3. **Matrix exponential calculation.** Now, the decomposition $C = M + N$ can be written as :

$$C = S\begin{pmatrix} \Lambda_A & O \\ O & \Lambda_B \end{pmatrix} S^{-1} + S\begin{pmatrix} O & H \\ O & O \end{pmatrix} S^{-1} \tag{3.11}$$

while M, N commute. Using this decomposition, we can calculate e^{Ct} as $e^{Mt}e^{Nt}$. First the exact value of e^{Mt} is :

$$\begin{aligned} e^{Mt} &= S \exp\left[\begin{pmatrix} \Lambda_A & O \\ O & \Lambda_B \end{pmatrix} t\right] S^{-1} = S \begin{pmatrix} e^{\Lambda_A t} & O \\ O & e^{\Lambda_B t} \end{pmatrix} S^{-1} \\ &= \begin{pmatrix} U_A e^{\Lambda_A t} & R e^{\Lambda_B t} \\ O & U_B e^{\Lambda_B t} \end{pmatrix} S^{-1} = \begin{pmatrix} U_A e^{\Lambda_A t} U_A^{-1} & U_A e^{\Lambda_A t}\left(-U_A^{-1} R U_B^{-1}\right) + R e^{\Lambda_B t} U_B^{-1} \\ O & U_B e^{\Lambda_B t} U_B^{-1} \end{pmatrix} \\ &= \begin{pmatrix} e^{At} & R U_B^{-1} e^{Bt} - e^{At} R U_B^{-1} \\ O & e^{Bt} \end{pmatrix}. \end{aligned}$$

While e^{Nt} has the exact value of :

$$\begin{aligned} e^{Nt} &= S \exp\left[\begin{pmatrix} O & H \\ O & O \end{pmatrix} t\right] S^{-1} = S \begin{pmatrix} I & Ht \\ O & I \end{pmatrix} S^{-1} \\ &= \begin{pmatrix} U_A & U_A H t + R \\ O & U_B \end{pmatrix} S^{-1} = \begin{pmatrix} I & -RU_B^{-1} + U_A H U_B^{-1} t + RU_B^{-1} \\ O & I \end{pmatrix} \\ &= \begin{pmatrix} I & U_A H U_B^{-1} t \\ O & I \end{pmatrix}. \end{aligned}$$

Thus, $F(t)$ from (3.8) is :

$$F(t) = e^{At} U_A H U_B^{-1} t + RU_B^{-1} e^{Bt} - e^{At} RU_B^{-1}$$

which induces the following integration result.

$$\int^t e^{-As} V e^{Bs} ds = U_A H U_B^{-1} t + e^{-At} RU_B^{-1} e^{Bt} - RU_B^{-1}. \tag{3.12}$$

With the result above, let's rewrite the terms in the forms directly involving the matrices A and B.

Lemma 3.1. *When A and B are non-singular real skew-symmetric matrices with V is a real matrix with rank 1,* (3.12) *holds :*

$$\int^t e^{-As} V e^{Bs} ds = U_A H U_B^{-1} t + e^{-At} RU_B^{-1} e^{Bt}.$$

When A and B do not share any eigenvalues,

$$\int^t e^{-As} V e^{Bs} ds = e^{-At} RU_B^{-1} e^{Bt}.$$

Proof. From (3.12), the last term can be erased since it is only a constant term in indefinite integral. If A, B do not share any eigenvalues, the first term vanishes since $H = 0$. □

Lemma 3.2. *When A, B are non-singular real skew-symmetric matrices and $V = ab^T$,*

$$\lim_{t\to\infty} \frac{1}{t} \int^t e^{-As} V e^{Bs} ds = \sum_{\lambda;\det(C-\lambda I)=0} P_\lambda^A a \left(P_\lambda^B b\right)^\dagger, \tag{3.13}$$

where P_λ^A represents a projection on eigenspace of A with eigenvalue λ.

Proof. Since e^{At} and e^{Bt} are orthogonal matrices, from (3.12),

$$\lim_{t\to\infty} \frac{1}{t} \int^t e^{-As} V e^{Bs} ds = U_A H U_B^{-1}.$$

First, from A being diagonalizable, $P_\lambda^A((A-\lambda I)q_\lambda) = 0$. With $(A-\lambda I)q_\lambda + a \in E_\lambda^A$, we can conclude that $(A-\lambda I)q_\lambda + a = P_\lambda^A a$. In case a is in column space of $A - \lambda I$, $P_\lambda^A a = 0$. Then, $\sum_{j;(B-\lambda I)v_j=0} b^T v_j v_j^\dagger = \left(P_\lambda^B b\right)^\dagger$. In case b is in row space of $B - \lambda I$, $P_\lambda^B b = 0$.

Combining these two results,

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{t} \int^t e^{-As} V e^{Bs} ds &= U_A H U_B^{-1} = U_A H U_B^\dagger \\ &= \sum_{\lambda;\det(C-\lambda I)} P_\lambda^A a \sum_{j;(B-\lambda I)v_j=0} b^T v_j v_j^\dagger = \sum_{\lambda;\det(C-\lambda I)} P_\lambda^A a \left(P_\lambda^B b\right)^\dagger. \end{aligned} \tag{3.14}$$

□

Additionally, if one of A or B does not have λ as an eigenvalue, its eigenspace is zero. Thus, we can use $\sum_{\lambda\in\mathbb{C}} P_\lambda^A a \left(P_\lambda^B b\right)^\dagger$ alternatively. Remark that if λ yields nonzero matrix of $P_\lambda^A a \left(P_\lambda^B b\right)^\dagger$, then $-\lambda$ also yields nonzero matrix since both A, B are skew-symmetric.

As a corollary of the lemma 3.2, when A and B do not share any eigenvalues, we get the following result.

Lemma 3.3. *When A, B are non-singular real skew-symmetric matrices with no common eigenvalues, for any real matrix V,*

$$\lim_{t\to\infty} \frac{1}{t}\int^t e^{-As} V e^{Bs} ds = 0. \tag{3.15}$$

Proof. We may decompose any matrix V into a sum of rank 1 matrices. Thus, assume that V is of rank 1 and prove the lemma. Note that at least one of $P_\lambda^A a$ or $P_\lambda^B b$ are zero, since λ with nonzero $P_\lambda^A a$ and $P_\lambda^B b$ is a common eigenvalue. Thus, from the lemma 3.2, we conclude the proof. □

3.4. **Example: nonsingular matrices with size 2.** For the special case, let's check when $A, B \in R^{2\times 2}$, A, B and $A - B$ are non-singular. Since A, B are skew-symmetric,

$$A = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} = U \begin{pmatrix} -ia & 0 \\ 0 & ia \end{pmatrix} U^\dagger, B = \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix} = U \begin{pmatrix} -ib & 0 \\ 0 & ib \end{pmatrix} U^\dagger, U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$$

Also, A and B do not share their eigenvalues. Thus, $H = 0$ and

$$\begin{aligned} R &= -\left(\tfrac{1}{a^2-b^2} \begin{pmatrix} ib & a \\ -a & ib \end{pmatrix} V \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \tfrac{1}{a^2-b^2} \begin{pmatrix} -ib & a \\ -a & -ib \end{pmatrix} V \tfrac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix} \right) \\ &= \frac{1}{a^2-b^2} \left\{ AVU + ibV \begin{pmatrix} i & -i \\ 1 & -1 \end{pmatrix} \right\}. \end{aligned}$$

Since

$$RU^\dagger = \frac{1}{a^2-b^2}\left\{AV + VB\right\},$$

from the lemma 3.1,

$$\int^t e^{-As} V e^{Bs} ds = \frac{1}{a^2-b^2}\left\{Ae^{-At}Ve^{Bt} + e^{-At}Ve^{Bt}B\right\}. \tag{3.16}$$

4. RG equation of Lohe sphere model with clustering conditions

In this section, we will focus on the clustering or complete synchronization in the Lohe sphere model. In detail, we aim to show that when there are groups of particles with identical natural frequencies, particles in the same group are likely to converge into a single particle in sense that such state forms a stable manifold.

4.1. **The clustering condition.** In this subsection we will define the clustering condition for the Lohe sphere model. The clustering condition represents when there are fixed choices of natural frequencies and each particle chooses its natural frequency among them. In the work of Y. Choi, S. Ha, and S. Yun [5], it is previously shown that every particles converges to a single particle if their natural frequencies are identical. The Lohe sphere model with clustering condition is written as :

$$\dot{x_{ij}} = \Omega_i x_{ij} + \frac{\kappa}{N}\left(I - x_{ij}x_{ij}^T\right)\sum_{k=1}^{m}\sum_{l=1}^{n_k} x_{kl}, \qquad N = \sum_{k=1}^{m} n_k. \tag{4.17}$$

Each Ω_is are skew-symmetric that satisfy the additional condition :

- Ω_i and Ω_j do not share any eigenvalues whenever $i \neq j$.

4.2. **Computation of the first order RG equation.** The Lohe sphere model with clustering condition has the system :

$$\dot{x_{ij}} = \Omega_i x_{ij} + \frac{\kappa}{N}\left(I - x_{ij}x_{ij}^T\right)\sum_{k=1}^{m}\sum_{l=1}^{n_k} x_{kl}, \qquad N = \sum_{k=1}^{m} n_k.$$

Define $\Omega = \mathrm{diag}(I_{n_1}\otimes\Omega_1, I_{n_2}\otimes\Omega_2, ..., I_{n_m}\otimes\Omega_m)$ as a block matrix. Call κ/N as ϵ. Then, the Lohe sphere model with the clustering condition is equivalent to :

$$\dot{x} = \Omega x + \epsilon g(x),$$

where $g(x)|_{ij} = \left(I - x_{ij}x_{ij}^T\right)\sum_{k=1}^{m}\sum_{l=1}^{n_k} x_{kl}$ and $x \in \mathbb{R}^{N(d+1)}$ is a concatenation of $x_{11}, x_{12}, ..., x_{mn_m}$. Note that,

$$X(s) = e^{\Omega s} = \mathrm{diag}(I_{n_1}\otimes e^{\Omega_1 s}, I_{n_2}\otimes e^{\Omega_2 s}, ..., I_{n_m}\otimes e^{\Omega_m s}).$$

From $G_1(x^{(0)}) := g(x^{(0)})$ with $x^{(0)} = X(s)y$,

$$\begin{aligned} G_1(X(s)y)|_{ij} &= \left(I_d - e^{\Omega_i s}y_{ij}y_{ij}^T e^{-\Omega_i s}\right)\sum_{k=1}^{m}\sum_{l=1}^{n_k} e^{\Omega_k s}y_{kl} \\ &= e^{\Omega_i s}\left(I - y_{ij}y_{ij}^T\right)\sum_{l=1}^{n_i} y_{il} + e^{\Omega_i s}\left(I - y_{ij}y_{ij}^T\right)\sum_{k\neq i}^{m}\sum_{l=1}^{n_k} e^{-\Omega_i s}e^{\Omega_k s}y_{kl}. \end{aligned}$$

For the remark, the natural frequencies do not commute, hence the equation $e^{\Omega_i}e^{\Omega_j} = e^{\Omega_i+\Omega_j}$ does not hold. Now, multiply both sides with $X(s)^{-1}$,

$$\begin{aligned} [X(s)^{-1}G_1(X(s)y)]_{ij} &= e^{-\Omega_i s}G_1(X(s)y)|_{ij} \\ &= \left(I - y_{ij}y_{ij}^T\right)\sum_{l=1}^{n_i} y_{il} + \left(I - y_{ij}y_{ij}^T\right)\sum_{k\neq i}^{m}\sum_{l=1}^{n_k} e^{-\Omega_i s}e^{\Omega_k s}y_{kl}. \end{aligned}$$

By integration,

$$\begin{aligned} &\left[\int^t X(s)^{-1}G_1(X(s)y)ds\right]_{ij} \\ &= \left(I - y_{ij}y_{ij}^T\right)\sum_{l=1}^{n_i} y_{il}t + \left(I - y_{ij}y_{ij}^T\right)\sum_{k\neq i}^{m}\sum_{l=1}^{n_k}\left[\int^t e^{-\Omega_i s}e^{\Omega_k s}ds\right]y_{kl} \end{aligned}$$

From the clustering condition, when $k \neq i$, Ω_i and Ω_k do not have identical eigenvalues. Thus, by the lemma 3.3,

$$\lim_{t\to\infty} \frac{1}{t} \int^t e^{-\Omega_i s} e^{\Omega_k s} ds = 0.$$

From this integration, we can build 1st order RG equation by calculating $R_1(y)$:

$$R_1(y)|_{ij} = \lim_{t\to\infty} \frac{1}{t} \left[\int^t X(s)^{-1} G_1(X(s)y) ds \right]_{ij} = \left(I - y_{ij} y_{ij}^T\right) \sum_{l=1}^{n_i} y_{il}. \tag{4.18}$$

Thus, 1st order RG equation is,

$$\dot{y_{ij}} = \epsilon R_1(y)|_{ij} = \frac{\kappa}{N} \left(I - y_{ij} y_{ij}^T\right) \sum_{l=1}^{n_i} y_{il}. \tag{4.19}$$

4.3. **Computation of the stable manifold.** For simplification, omit the subscript i from the 1st order RG equation (4.19) since particles are independent if they have different natural frequency matrix.

$$\dot{y_j} = \epsilon \left(I - y_j y_j^T\right) \sum_{l=1}^{n} y_l. \tag{4.20}$$

The equation (4.20) has a form identical to the Lohe sphere model when natural frequencies of each particle are identical. From the work of the Y. Choi, S. Ha, and S. Yun [5], we already know that such system converges to the state where each particles are in same position. Hence, the invariant manifold

$$M := \{x_{i1} = x_{i2} = \cdots = x_{in_i}, i = 1, 2, \cdots, m\},$$

which represents the complete synchronization among particles with same natural frequencies, is stable. Note that the manifold M of the 1st order RG system (4.19) corresponds to the same manifold M in the original system (4.17). Thus, by the symmetry of the stable manifolds, the theorem 2.2, the invariant manifold M of (4.17) is also stable in the original system (4.17).

Theorem 4.1. *When Lohe sphere model with groups of particles with identical natural frequencies :*

$$\dot{x_{ij}} = \Omega_i x_{ij} + \frac{\kappa}{N} \left(I - x_{ij} x_{ij}^T\right) \sum_{k=1}^{m} \sum_{l=1}^{n_k} x_{kl}, \qquad N = \sum_{k=1}^{m} n_k,$$

satisfies the condition where Ω_i and Ω_j do not share any eigenvalues whenever $i \neq j$, then for sufficiently small κ, the invariant manifold M,

$$M := \{x_{i1} = x_{i2} = \cdots = x_{in_i}, i = 1, 2, \cdots, m\},$$

is a stable manifold of the system.

While the complete synchronization among identical natural frequencies forms a stable manifold, it is uncertain whether the system actually converges to M. Since the prior works on the symmetry between invariant manifolds only ensures the stability, it is not possible to conclude the actual convergence. However, current theorems on RG method includes the general form of ODEs with perturbations. It might be possible to obtain symmetries in convergence itself, if we specify the target, rather than handling general ODE. We leave the question for the future work.

5. Simulations

In this section, we performed numerical simulations and it coincides with the results from the section 4. Runge-Kutta method of order 4 is used. We considered the Lohe sphere model with dimension 2, *i.e.* the Kuramoto model. Group of $m = 5$ with $\kappa = 0.05$ were used, with $(n_i, \omega_i) = (7, 0.05), (12, 0.07), (11, 0.08), (9, 0.03), (11, 0.02)$.

The Figure 1 shows the angular position of every 50 particles by time. Slopes in the figure represent particles' angular velocity. We could see that the slopes converges to the same value, representing the synchronization phenomenon. The Figure 3 shows the angular position in circle, *i.e.* modulus by 2π, for particles in each groups. We could observe that particles in the same group shows clustering phenomenon or complete synchronization, coinciding with the stable manifold we computed in the section 4. However, it seems the clustering phenomenon are present only for the particles in the same group, as shown in The Figure 2.

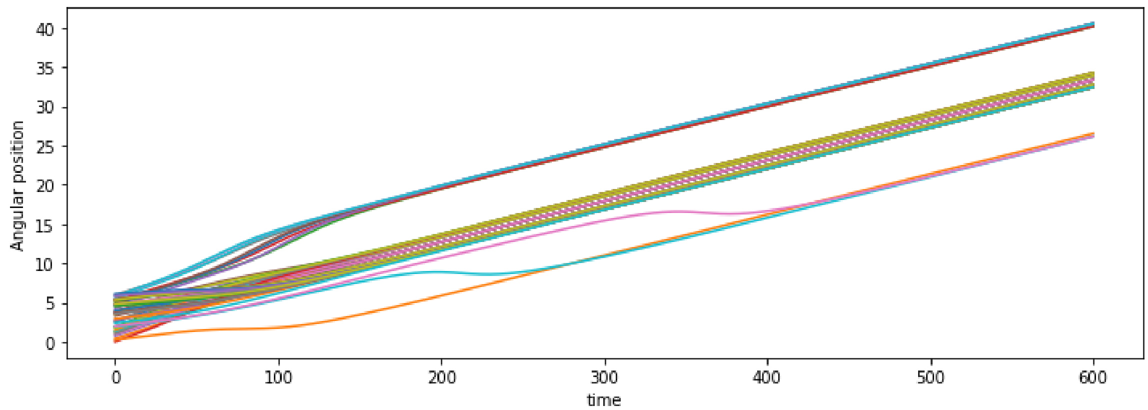

Figure 1. Angular position of every 50 particles by time. As the time flows, we can check the synchronization phenomenon.

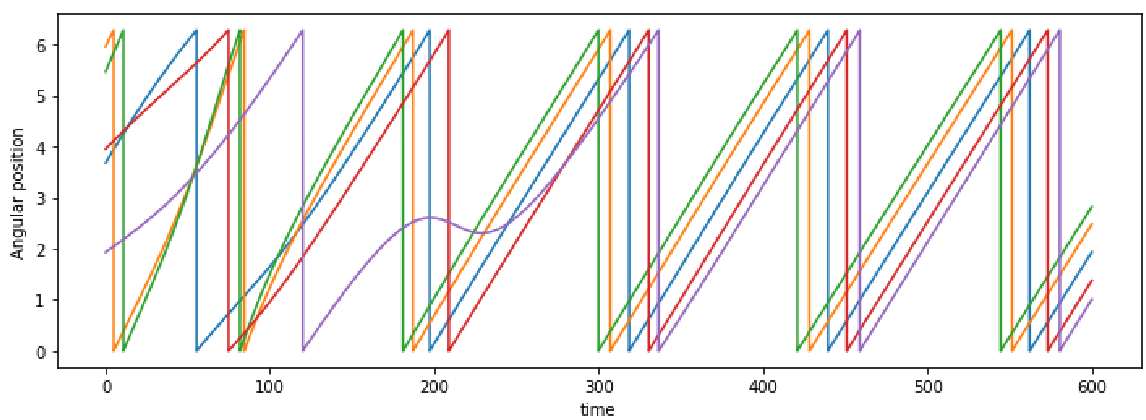

Figure 2. Angular position of 5 particles with different natural frequencies. We can check the synchronization phenomenon but they do not cluster as one particle.

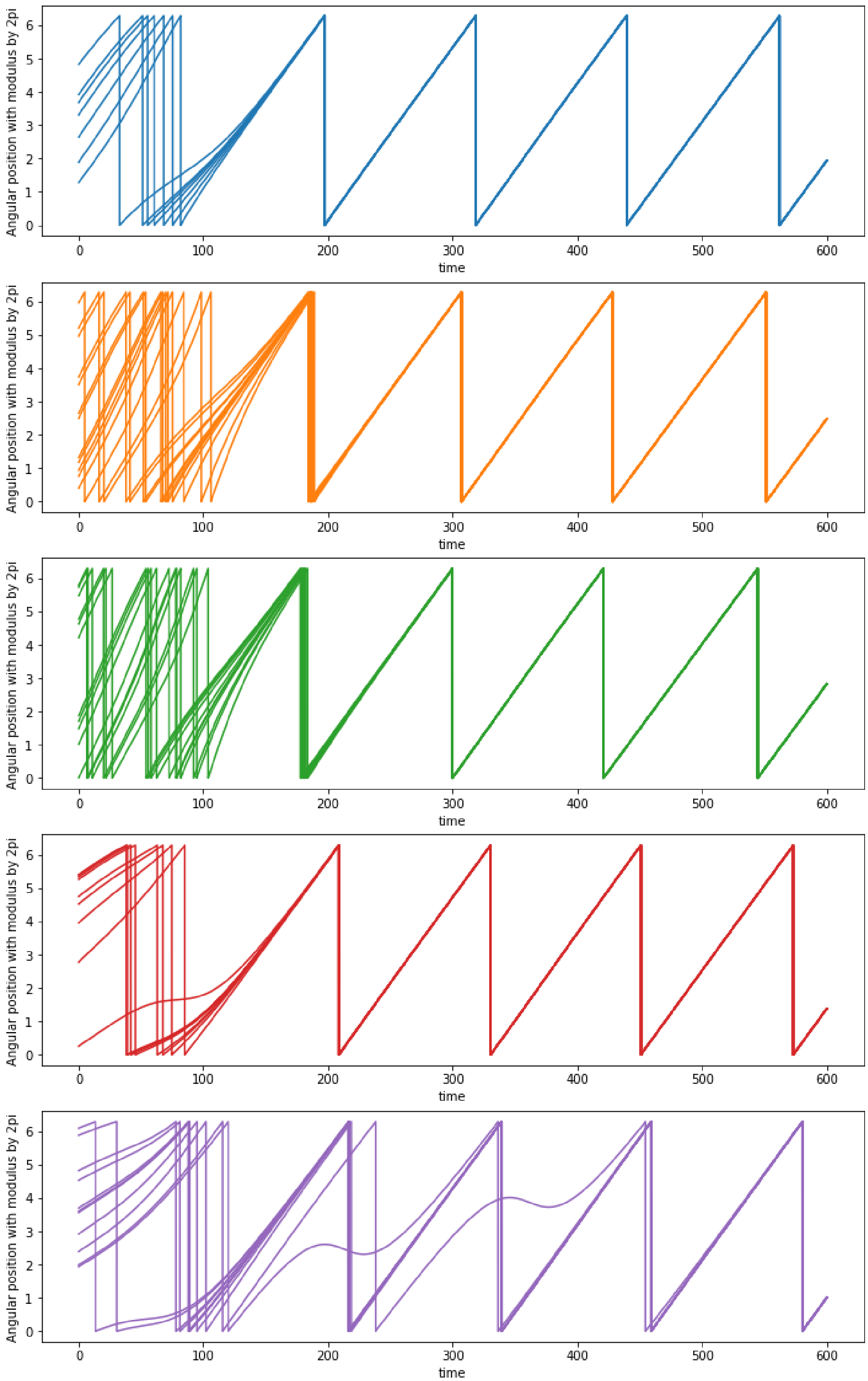

FIGURE 3. Angular position (mod 2π) of every 50 particles grouped by their natural frequencies. We can check that each group converges to a single particle as the time goes.

6. RG equation of Lohe sphere model with non-resonance condition

In this section, we will consider the Lohe sphere model with the non-resonance condition and compute its RG equation up to 2nd order. The non-resonance condition excludes the situations where the particles' natural frequencies coincide.

6.1. **The non-resonance condition.** For computational simplicity, we will give additional conditions that represents a non-resonance situation on the Lohe sphere model. The non-resonance conditions are defined as :

(1) $\Omega_i\Omega_j$ is symmetric for any i, j. In other words, natural frequencies Ω_i and Ω_j commutes in matrix multiplication. Additionally, $e^{\Omega_i}e^{\Omega_j} = e^{\Omega_i+\Omega_j}$ holds only when this condition holds.
(2) $\Omega_i - \Omega_j$ is nonsingular whenever $i \neq j$.
(3) $\Omega_i - \Omega_j$ and $\Omega_j - \Omega_k$ do not share any eigenvalues whenever i, j, k are distinct indices.

In 2-dimension, every skew-symmetric matrices are in form of

$$\Omega = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} = \omega \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \omega \in \mathbb{R}.$$

If we rewrite the 2-dimension Lohe sphere model, it is equivalent to the Kuramoto model. The equivalent non-resonance conditions of the Kuramoto model is :

(1) $\omega_i - \omega_j$ is non-zero whenever $i \neq j$.
(2) $\omega_i - \omega_j \neq \omega_j - \omega_k$ whenever i, j, k are distinct indicies.

6.2. **First order RG equation of Lohe sphere model.** The inhomogeneous term G_1 is given by

$$G_1(x^{(0)}) := g(x^{(0)}).$$

From the definition $x^{(0)} = X(s)y$,

$$G_1(X(s)y)|_i = \left(I - e^{\Omega_i s}y_i y_i^T e^{-\Omega_i s}\right) \sum_{k=1}^{N} e^{\Omega_k s} y_k.$$

Since $X(s)^{-1} = \operatorname{diag}(e^{-\Omega_1 s}, e^{-\Omega_2 s}, ..., e^{-\Omega_N s})$,

$$\begin{aligned} [X(s)^{-1}G_1(X(s)y)]_i &= e^{-\Omega_i s}\left(I - e^{\Omega_i s}y_i y_i^T e^{-\Omega_i s}\right) \sum_{k=1}^{N} e^{\Omega_k s} y_k \\ &= \left(e^{-\Omega_i s} - y_i y_i^T e^{-\Omega_i s}\right) \sum_{k=1}^{N} e^{\Omega_k s} y_k \\ &= \left(I - y_i y_i^T\right) \sum_{k=1}^{N} e^{(\Omega_k - \Omega_i)s} y_k. \end{aligned}$$

Now, if we integrate it,

$$\left[\int^t X(s)^{-1}G_1(X(s)y)ds\right]_i = \left(I - y_i y_i^T\right)\left[y_i t + \sum_{k\neq i} \int^t e^{(\Omega_k - \Omega_i)s} y_k ds\right]. \tag{6.21}$$

From this integration, we can build 1st order RG equation by calculating $R_1(y)$ and $h_t^{(1)}(y)$. Since $e^{(\Omega_k-\Omega_i)t}$ is an orthogonal matrix, thus being bounded,

$$\begin{aligned} R_1(y)|_i &= \lim_{t\to\infty}\frac{1}{t}\left[\int^t X(s)^{-1}G_1(X(s)y)ds\right]_i \\ &= \left(I - y_i y_i^T\right) y_i \\ &= y_i\left(1 - y_i^T y_i\right). \end{aligned} \tag{6.22}$$

Furthermore, $h_t^{(1)}(y)$ is calculated as :

$$\begin{aligned} h_t^{(1)}(y)|_i &= X(t)\left[\int^t X(s)^{-1}G_1(X(s)y) - R_1(y)ds\right]_i \\ &= e^{\Omega_i t}\left(I - y_i y_i^T\right)\left[\sum_{k\neq i}^N \left(\Omega_k-\Omega_i\right)^{-1} e^{(\Omega_k-\Omega_i)t} y_k\right] \\ &= \sum_{k\neq i}^N \left(\Omega_k-\Omega_i\right)^{-1} e^{\Omega_k t} y_k - e^{\Omega_i t} y_i \sum_{k\neq i}^N \left\langle y_i, \left(\Omega_k-\Omega_i\right)^{-1} e^{(\Omega_k-\Omega_i)t} y_k\right\rangle. \end{aligned} \tag{6.23}$$

Thus, 1st order RG equation is,

$$\dot{y_i} = \epsilon R_1(y)|_i = \frac{\kappa}{N} y_i\left(1 - ||y_i||^2\right). \tag{6.24}$$

Since $y(0) = X(0)^{-1}x(0)$, $||y_i(0)|| = 1$. Thus, y stays constant.

The 1st order approximation $\tilde{x} = X(t)y(t) + \epsilon h_t^{(1)}(y)$ of x is :

$$\tilde{x}(t)|_i = e^{\Omega_i t} y_i + \epsilon e^{\Omega_i t}\left(I - y_i y_i^T\right)\left[\sum_{k\neq i}^N \left(\Omega_k-\Omega_i\right)^{-1} e^{(\Omega_k-\Omega_i)t} y_k\right]. \tag{6.25}$$

We can easily check 1st order approximation also lies on the same sphere. However, since 1st order RG equation is zero, further calculation for higher order is needed for symmetry of the stable manifold.

6.3. **Second order RG equation of Lohe sphere model.** First, the term $\frac{dg}{dx}$ is :

$$\frac{dg_i}{dx_j} = \begin{cases} I - x_i x_i^T, & i \neq j \\ (I - x_i x_i^T) - (x_i^T \sum_{k=1}^N x_k) I - x_i \sum_{k=1}^N x_k^T, & i = j. \end{cases}$$

The second order term of inhomogeneous part $G_2(x^{(0)}, x^{(1)}) := \frac{dg}{dx}(x^{(0)})x^{(1)}$ is :

$$\begin{aligned} &G_2(x^{(0)}, x^{(1)})|_i \\ &= \left(I - x_i^{(0)} x_i^{(0)T}\right)\sum_{k=1}^N x_k^{(1)} - \left(x_i^{(1)} x_i^{(0)T} + x_i^{(0)} x_i^{(1)T}\right)\sum_{k=1}^N x_k^{(0)} \\ &= \sum_{k=1}^N x_k^{(1)} - x_i^{(0)}\sum_{k=1}^N\left\langle x_i^{(0)}, x_k^{(1)}\right\rangle - x_i^{(1)}\sum_{k=1}^N\left\langle x_i^{(0)}, x_k^{(0)}\right\rangle - x_i^{(0)}\sum_{k=1}^N\left\langle x_i^{(1)}, x_k^{(0)}\right\rangle. \end{aligned}$$

By substituting the results from first order computation,

$$
\begin{aligned}
&G_2(X(s)y, h_s^{(1)}(y))|_i \\
&\quad= \sum_{j=1}^{N}\sum_{k\neq j}^{N} \left[(\Omega_k-\Omega_j)^{-1}e^{\Omega_k s}y_k - e^{\Omega_j s}y_j\left\langle y_j, (\Omega_k-\Omega_j)^{-1}e^{(\Omega_k-\Omega_j)s}y_k\right\rangle\right] \\
&\quad- \sum_{j=1}^{N}\sum_{k\neq j}^{N} e^{\Omega_i s}y_i\left\langle e^{\Omega_i s}y_i, (\Omega_k-\Omega_j)^{-1}e^{\Omega_k s}y_k - e^{\Omega_j s}y_j\left\langle y_j, (\Omega_k-\Omega_j)^{-1}e^{(\Omega_k-\Omega_j)s}y_k\right\rangle\right\rangle \\
&\quad- \sum_{k\neq i}^{N}\left[(\Omega_k-\Omega_i)^{-1}e^{\Omega_k s}y_k - e^{\Omega_i s}y_i\left\langle y_i, (\Omega_k-\Omega_i)^{-1}e^{(\Omega_k-\Omega_i)s}y_k\right\rangle\right]\sum_{j=1}^{N}\left\langle e^{\Omega_i s}y_i, e^{\Omega_j s}y_j\right\rangle \\
&\quad- \sum_{j=1}^{N}\sum_{k\neq i}^{N} e^{\Omega_i s}y_i\left\langle e^{\Omega_j s}y_j, (\Omega_k-\Omega_i)^{-1}e^{\Omega_k s}y_k - e^{\Omega_i s}y_i\left\langle y_i, (\Omega_k-\Omega_i)^{-1}e^{(\Omega_k-\Omega_i)s}y_k\right\rangle\right\rangle,
\end{aligned}
$$

which can be simplified as :

$$
\begin{aligned}
&X(s)^{-1}G_2(X(s)y, h_s^{(1)}(y))|_i \\
&\quad= \sum_{j=1}^{N}\sum_{k\neq j}^{N}\left[(\Omega_k-\Omega_j)^{-1}e^{(\Omega_k-\Omega_i)s}y_k - e^{(\Omega_j-\Omega_i)s}y_jy_j^T(\Omega_k-\Omega_j)^{-1}e^{(\Omega_k-\Omega_j)s}y_k\right] \\
&\quad- \sum_{j=1}^{N}\sum_{k\neq j}^{N} y_iy_i^T\left[(\Omega_k-\Omega_j)^{-1}e^{(\Omega_k-\Omega_i)s}y_k - e^{(\Omega_j-\Omega_i)s}y_jy_j^T(\Omega_k-\Omega_j)^{-1}e^{(\Omega_k-\Omega_j)s}y_k\right] \\
&\quad- \sum_{j=1}^{N}\sum_{k\neq i}^{N}\left(I-y_iy_i^T\right)(\Omega_k-\Omega_i)^{-1}e^{(\Omega_k-\Omega_i)s}y_ky_j^Te^{(\Omega_i-\Omega_j)s}y_i \\
&\quad- \sum_{j=1}^{N}\sum_{k\neq i}^{N} y_iy_j^T\left[(\Omega_k-\Omega_i)^{-1}e^{(\Omega_k-\Omega_j)s}y_k - e^{(\Omega_i-\Omega_j)s}y_iy_i^T(\Omega_k-\Omega_i)^{-1}e^{(\Omega_k-\Omega_i)s}y_k\right].
\end{aligned}
$$

To compute $\left[\int^t X(s)^{-1}G_2(X(s)y, h_s^{(1)}(y))ds\right]_i$, separate the summation into the four cases : 1.$(i\neq j, j\neq k, k\neq i)$, 2.$(j\neq i, k=i)$, 3.$(k\neq i, j=i)$, 4.$(k=j\neq i)$. Integration by parts with the help of the fact that $x^TAx=0$ for any skew-symmetric A gives the integration as :

$$
\begin{aligned}
&\left[\int^t X(s)^{-1}G_2(X(s)y, h_s^{(1)}(y))ds\right]_i \\
&= \sum_{j\neq i}\left[(\Omega_i-\Omega_j)^{-1} y_i t + \frac{1}{2} y_i \left\langle y_j, (\Omega_i-\Omega_j)^{-1} e^{(\Omega_i-\Omega_j)t} y_i\right\rangle^2\right] \\
&- \sum_{j\neq i}\left(I - y_i y_i^T\right)(\Omega_j-\Omega_i)^{-1} e^{-(\Omega_i-\Omega_j)t} y_j y_j^T (\Omega_j-\Omega_i)^{-1} e^{(\Omega_i-\Omega_j)t} y_i \\
&+ \sum_{j\neq i}\left(1 - y_i^T y_i\right)\left(I - 3y_i y_i^T\right)(\Omega_j-\Omega_i)^{-2} e^{(\Omega_j-\Omega_i)t} y_j \\
&+ \sum_{i\neq j, j\neq k, k\neq i}\left(I - y_i y_i^T\right)(\Omega_k-\Omega_j)^{-1}(\Omega_k-\Omega_i)^{-1} e^{(\Omega_k-\Omega_i)t} y_k \\
&- \sum_{i\neq j, j\neq k, k\neq i} y_i y_j^T (\Omega_k-\Omega_j)^{-1}(\Omega_k-\Omega_i)^{-1} e^{(\Omega_k-\Omega_j)t} y_k \\
&+ \sum_{i\neq j, j\neq k, k\neq i} y_i \left\langle y_j, (\Omega_i-\Omega_j)^{-1} e^{(\Omega_i-\Omega_j)t} y_i\right\rangle \left\langle y_k, (\Omega_i-\Omega_k)^{-1} e^{(\Omega_i-\Omega_k)t} y_i\right\rangle \\
&- \sum_{i\neq j, j\neq k, k\neq i} \int^t (\Omega_k-\Omega_i)^{-1} e^{-(\Omega_i-\Omega_k)s} y_k y_j^T e^{(\Omega_i-\Omega_j)s} y_i ds \\
&- \sum_{i\neq j, j\neq k, k\neq i} \left(I - y_i y_i^T\right) \int^t e^{-(\Omega_i-\Omega_j)s} y_j y_j^T (\Omega_k-\Omega_j)^{-1} e^{(\Omega_k-\Omega_j)s} y_k ds
\end{aligned}
$$

From the Lemma 3.2, $\lim_{t\to\infty}\frac{1}{t}\int^t e^{-As}Ve^{-Bs}ds = 0$ if A, B don't have any common eigenvalues. The matrix exponential e^{At} is unitary if A is skew-symmetric, hence bounded. Thus,

$$
\lim_{t\to\infty}\frac{1}{t}\left[\int^t X(s)^{-1}G_2(X(s)y, h_s^{(1)}(y))ds\right]_i = \sum_{j\neq i}^{N} (\Omega_i-\Omega_j)^{-1} y_i.
$$

Additionally, from

$$
\frac{\partial h_t^{(1)}(y)|_i}{\partial y_j} = \begin{cases} (\Omega_j-\Omega_i)^{-1} e^{\Omega_j t} - e^{\Omega_i t} y_i y_i^T (\Omega_j-\Omega_i)^{-1} e^{(\Omega_j-\Omega_i)t} & (i\neq j) \\ e^{\Omega_i t}\sum_{k\neq i}^N \left\langle y_k, e^{-(\Omega_k-\Omega_i)t}(\Omega_k-\Omega_i)^{-1} y_i\right\rangle + e^{\Omega_i t} y_i \sum_{k\neq i}^N y_k^T e^{-(\Omega_k-\Omega_i)t}(\Omega_k-\Omega_i)^{-1} & (i=j), \end{cases}
$$

we have

$$
\begin{aligned}
&X(s)^{-1}\left(Dh_s^1\right)_y R_1(y)|_i \\
&= y_i\left(1 - y_i^T y_i\right)\sum_{k\neq i}^N \left\langle y_k, e^{-(\Omega_k-\Omega_i)s}(\Omega_k-\Omega_i)^{-1} y_i\right\rangle \\
&+ y_i \sum_{k\neq i}^N y_k^T e^{-(\Omega_k-\Omega_i)s}(\Omega_k-\Omega_i)^{-1} y_i \left(1 - y_i^T y_i\right) \\
&+ \sum_{j\neq i}\left(I - y_i y_i^T\right)(\Omega_j-\Omega_i)^{-1} e^{(\Omega_j-\Omega_i)s} y_j \left(1 - y_j^T y_j\right).
\end{aligned}
$$

This yields :

$$\begin{aligned}&\left[\int^t X(s)^{-1}\left(Dh_s^1\right)_y R_1(y)ds\right]_i\\&= -y_i\left(1-y_i^T y_i\right)\sum_{k\neq i}^N\left\langle y_k, e^{-(\Omega_k-\Omega_i)t}\left(\Omega_k-\Omega_i\right)^{-2} y_i\right\rangle\\&- y_i\sum_{k\neq i}^N y_k^T e^{-(\Omega_k-\Omega_i)t}\left(\Omega_k-\Omega_i\right)^{-2} y_i\left(1-y_i^T y_i\right)\\&-\sum_{j\neq i}\left(I-y_i y_i^T\right)\left(\Omega_j-\Omega_i\right)^{-2} e^{(\Omega_j-\Omega_i)t} y_j\left(1-y_j^T y_j\right).\end{aligned}$$

Hence,

$$\lim_{t\to\infty}\left[\frac{1}{t}\int^t X(s)^{-1}\left(Dh_s^1\right)_y R_1(y)ds\right]_i = 0.$$

Thus, R_2 is :

$$R_2(y)|_i = \sum_{k\neq i}^N\left(\Omega_i-\Omega_k\right)^{-1} y_i. \tag{6.26}$$

While R_2 is indeed a non-zero, it is impossible to obtain any useful results regarding on the stable manifold. The 2nd order RG equation shows particles on the fixed orbit, independent to each other. To obtain the stable invariant manifold of the original Lohe model, we need to find a stable invariant manifold of the 2nd RG equation, which seems impossible in this situation.

6.4. **2nd order RG approximate solution for Kuramoto model.** In this subsection, we restrict $d = 1$ and compute the approximate solution for Kuramoto model using 2nd order RG equation. The 2nd order RG equation for Lohe model is :

$$\begin{aligned}\dot{y_i} &= \epsilon R_1(y)|_i + \epsilon^2 R_2(y)|_i\\&= \epsilon y_i\left(1-\|y_i\|^2\right) + \epsilon^2\sum_{k\neq i}^N\left(\Omega_i-\Omega_k\right)^{-1} y_i.\end{aligned}$$

With the y_i starting on a unit circle, each y_i always satisfies $\|y_i\| = 1$ at any time. When each y_i is $x_i(0)$ at time 0, the solution for the 2nd order RG equation is,

$$y_i(t) = e^{\epsilon^2\sum_{k\neq i}^N(\Omega_i-\Omega_k)^{-1}t} x_i(0).$$

We already know $h_t^{(1)}(y)$:

$$h_t^{(1)}(y)|_i = \sum_{k\neq i}^N\left(\Omega_k-\Omega_i\right)^{-1} e^{\Omega_k t} y_k - e^{\Omega_i t} y_i\sum_{k\neq i}^N\left\langle y_i, \left(\Omega_k-\Omega_i\right)^{-1} e^{(\Omega_k-\Omega_i)t} y_k\right\rangle.$$

Now compute $h_t^{(2)}(y)$ using $\|y_i\| = 1$ and the computation result in the section 3.4. When $\Omega_i = \omega_i\rho, \rho = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$,

$$
\begin{aligned}
&X(t)^{-1}h_t^{(2)}(y)|_i \\
&\quad = \frac{1}{2}y_i\left[\sum_{j\neq i}\left\langle y_j, (\Omega_i-\Omega_j)^{-1}e^{(\Omega_i-\Omega_j)t}y_i\right\rangle\right]^2 \\
&\quad - \sum_{j\neq i}\left(I-y_iy_i^T\right)(\Omega_j-\Omega_i)^{-1}e^{-(\Omega_i-\Omega_j)t}y_jy_j^T(\Omega_j-\Omega_i)^{-1}e^{(\Omega_i-\Omega_j)t}y_i \\
&\quad + \sum_{i\neq j, j\neq k, k\neq i}\left(I-y_iy_i^T\right)(\Omega_k-\Omega_j)^{-1}(\Omega_k-\Omega_i)^{-1}e^{(\Omega_k-\Omega_i)t}y_k \\
&\quad - \sum_{i\neq j, j\neq k, k\neq i} y_iy_j^T(\Omega_k-\Omega_j)^{-1}(\Omega_k-\Omega_i)^{-1}e^{(\Omega_k-\Omega_j)t}y_k \\
&\quad + \sum_{i\neq j, j\neq k, k\neq i}\frac{1}{(\omega_j-\omega_k)(2\omega_i-\omega_j-\omega_k)}e^{-(\Omega_i-\Omega_k)t}y_ky_j^Te^{(\Omega_i-\Omega_j)t}y_i \\
&\quad + \sum_{i\neq j, j\neq k, k\neq i}\frac{1}{(\omega_j-\omega_k)(2\omega_i-\omega_j-\omega_k)}(\Omega_i-\Omega_k)^{-1}e^{-(\Omega_i-\Omega_k)t}y_ky_j^Te^{(\Omega_i-\Omega_j)t}(\Omega_i-\Omega_j)y_i \\
&\quad + \sum_{i\neq j, j\neq k, k\neq i}\left(I-y_iy_i^T\right)\frac{1}{(\omega_i-\omega_k)(2\omega_j-\omega_i-\omega_k)}e^{-(\Omega_i-\Omega_j)t}y_jy_j^Te^{(\Omega_k-\Omega_j)t}y_k \\
&\quad + \sum_{i\neq j, j\neq k, k\neq i}\left(I-y_iy_i^T\right)\frac{1}{(\omega_i-\omega_k)(2\omega_j-\omega_i-\omega_k)}(\Omega_i-\Omega_j)e^{-(\Omega_i-\Omega_j)t}y_jy_j^T(\Omega_k-\Omega_j)^{-1}e^{(\Omega_k-\Omega_j)t}y_k.
\end{aligned}
$$

Let's define a new set of variables $z_i(t) := e^{\Omega_i t}y_i(t)$. Then,

$$
\tilde{h}_t^{(1)}(z(t))|_i = \sum_{j\neq i}^{N}(\omega_i-\omega_j)^{-1}\left[\rho z_j + z_i\langle z_j, \rho z_i\rangle\right],
$$

$$
\begin{aligned}
\tilde{h}_t^{(2)}(z)|_i = {}& \frac{1}{2}z_i\left[\sum_{j\neq i}(\omega_i-\omega_j)^{-1}\langle z_j,\rho z_i\rangle\right]^2 - \sum_{j\neq i}(\omega_j-\omega_i)^{-2}\left[\rho z_j\langle z_j,\rho z_i\rangle + z_i\langle z_j,\rho z_i\rangle^2\right] \\
& + \sum_{i\neq j, j\neq k, k\neq i}(\omega_k-\omega_j)^{-1}(\omega_k-\omega_i)^{-1}\left[z_i\langle z_i,z_k\rangle + z_i\langle z_j,z_k\rangle - z_k\right] \\
& + \sum_{i\neq j, j\neq k, k\neq i}\frac{1}{(\omega_j-\omega_k)(2\omega_i-\omega_j-\omega_k)}z_k\langle z_j,z_i\rangle \\
& - \sum_{i\neq j, j\neq k, k\neq i}\frac{(\omega_i-\omega_j)}{(\omega_j-\omega_k)(\omega_i-\omega_k)(2\omega_i-\omega_j-\omega_k)}\rho z_k\langle z_j,\rho z_i\rangle \\
& + \sum_{i\neq j, j\neq k, k\neq i}\frac{1}{(\omega_i-\omega_k)(2\omega_j-\omega_i-\omega_k)}\left[z_j - z_i\langle z_i,z_j\rangle\right]\langle z_j,z_k\rangle \\
& - \sum_{i\neq j, j\neq k, k\neq i}\frac{(\omega_i-\omega_j)}{(\omega_i-\omega_k)(\omega_k-\omega_j)(2\omega_j-\omega_i-\omega_k)}\left[\rho z_j + z_i\langle z_j,\rho z_i\rangle\right]\langle z_j,\rho z_k\rangle.
\end{aligned}
$$

Thus, the approximated solution using 2nd RG equation is

$$\tilde{x}(t)|_i = z_i(t) + \epsilon \tilde{h}_t^{(1)}(z)|_i + \epsilon^2 \tilde{h}_t^{(2)}(z)|_i,$$

where

$$z_i(t) = e^{\left[\omega_i - \epsilon^2 \sum_{k \neq i}^{N} (\omega_i - \omega_k)^{-1}\right]\rho t} x_i(0).$$

7. Conclusion

The Lohe sphere model or the Kuramoto model are mathematical models that describes the synchronization or clustering phenomenon. However, due to its complexity, the convergence analysis is a key question in these systems. In this paper, we applied the Renormalization Group method for the convergence analysis. The Renormalization Group method provides the technique to compute the approximated solution for the given differential equation system. The approximated solution by RG method does not contain the secular term, which is a term that anomaly diverges when the time goes to infinity. Such characteristic of RG method provides two advantages, a approximated solution for a longer time interval, and a symmetry of stability on invariant manifolds between RG system and original system.

With the clustering conditions, we were able to find the stable manifold using the first order RG equation. Generalizing the results from the work [5], we showed that the manifold corresponding to the clustering or complete synchronization on the groups of particles with identical natural frequency matrices is indeed the stable manifold. In the future studies, we plan to work on the symmetry of convergence radius of the stable manifolds, leading to the convergence conditions on the Lohe sphere model or the Kuramoto model using the convergence radius of the stable manifolds of RG equations.

We also provided the results of 2nd order RG method with the non-resonance conditions. While the 2nd order RG equation did not provide significant information about stable manifolds of the Lohe sphere model, we successfully computed the 2nd order RG approximated solution for the Kuramoto model. To analyze general synchronization behavior using stable manifolds of RG equation, it seems more conditions, rather than non-resonance condition, are required, focusing on specific instances of the Lohe sphere model. We defer such work in the future studies.

References

1. Hayato Chiba, *C1 approximation of vector fields based on the renormalization group method*, SIAM Journal on Applied Dynamical Systems **7** (2008), no. 3, 895–932 (English).
2. ______, *Renormalization group method and its application to coupled oscillators (mathematical aspects for nonlinear problems related to life-phenomena)*, (2008).
3. ______, *Simplified renormalization group method for ordinary differential equations*, Journal of Differential Equations **246** (2009), no. 5, 1991–2019.
4. Sun-Ho Choi and Seung-Yeal Ha, *Complete entrainment of lohe oscillators under attractive and repulsive couplings*, SIAM Journal on Applied Dynamical Systems **13** (2014), no. 4, 1417–1441.
5. Young-Pil Choi, Seung Yeal Ha, and Seok-Bae Yun, *Complete synchronization of kuramoto oscillators with finite inertia*, Physica D: Nonlinear Phenomena **240** (2011), 32–44.
6. D. Cumin and C.P. Unsworth, *Generalising the kuramoto model for the study of neuronal synchronisation in the brain*, Physica D: Nonlinear Phenomena **226** (2007), no. 2, 181–196 (English).
7. Seung Yeal Ha, Hwa Kil Kim, and Jinyeong Park, *Remarks on the complete synchronization for the kuramoto model with frustrations*, Analysis and Applications **16** (2017).
8. Yoshiki Kuramoto, *Self-entrainment of a population of coupled non-linear oscillators*, International Symposium on Mathematical Problems in Theoretical Physics (Berlin, Heidelberg) (Huzihiro Araki, ed.), Springer Berlin Heidelberg, 1975, pp. 420–422.

9. M A Lohe, *Non-abelian kuramoto models and synchronization*, Journal of Physics A: Mathematical and Theoretical **42** (2009), no. 39, 395101.
10. Tamás Vicsek, András Czirók, Eshel Ben-Jacob, Inon Cohen, and Ofer Shochet, *Novel type of phase transition in a system of self-driven particles*, Phys. Rev. Lett. **75** (1995), 1226–1229.

Appendices

Following is the code written for the simulations in section 5.

```
import numpy as np
import matplotlib.pyplot as plt

def Kuramoto_gen(N, kappa, omega) :
  def f(t, theta) :
    dtheta = np.zeros(N)
    for j in range(N) :
      dtheta += np.sin(theta[j] - theta)

    dtheta = omega + kappa / N * dtheta
    return dtheta
  # end f

  return f

N=50
kappa = 0.05
omega = np.concatenate(
  (0.05 * np.ones(7),
  0.07 * np.ones(12),
  0.08 * np.ones(11),
  0.03 * np.ones(9),
  0.02 * np.ones(11))
)

clustered = Kuramoto_gen(N, kappa, omega)

n = 40000
h = 0.01

# f : function, (x_0, y_0) : initial condition, step size h, iteration n
def RK4(f, t_0, theta_0, n, h) :
  t = [t_0]
  theta = [theta_0]

  for k in range(1, n+1) :
    t.append(t_0 + h * k)
```

```
    k_1 = f(t[k-1], theta[k-1])
    k_2 = f(t[k-1] + 0.5 * h, theta[k-1]+0.5*h*k_1)
    k_3 = f(t[k-1] + 0.5 * h, theta[k-1]+0.5*h*k_2)
    k_4 = f(t[k-1] + h, theta[k-1] + h * k_3)
    theta.append(theta[k-1] + h * (k_1 + 2* k_2 + 2* k_3 + k_4)/6)

  return t, theta
# end RK4

t, theta = RK4(clustered, 0, np.random.uniform(0, 5, N), n, h)

plt.figure(figsize=(12,4))
plt.plot(t, theta)
plt.xlabel("time")
plt.ylabel("Angular position")

theta = np.mod(theta, np.pi)

fig, axs = plt.subplots(5, figsize=(12,4 * 5))
axs[0].plot(t, theta[:, 0:7])
axs[1].plot(t, theta[:, 7:19])
axs[2].plot(t, theta[:, 19:30])
axs[3].plot(t, theta[:, 30:39])
axs[4].plot(t, theta[:, 39:50])

for ax in axs.flat:
    ax.set(xlabel='time', ylabel='Angular position with modulus by 2pi')

plt.figure(figsize=(12,4))
plt.plot(t, theta[:, [0,7,19, 30, 39]])
plt.xlabel("time")
plt.ylabel("Angular position")
```

(Jinhee Paeng)
Department of Mathematical Sciences
Seoul National University, Seoul 08826, Republic of Korea
Email address: jhpaeng306@snu.ac.kr

(Kibeom Myoung)
Department of Mathematical Sciences
Seoul National University, Seoul 08826, Republic of Korea
Email address: mkb0414@snu.ac.kr

(Seungmok Lee)
Department of Mathematical Sciences
Seoul National University, Seoul 08826, Republic of Korea
Email address: kozyrak@snu.ac.kr

(Seung Yeal Ha)
Department of Mathematical Sciences
Seoul National University, Seoul 08826, Republic of Korea
Email address: syha@snu.ac.kr

(Jaeyoung Yoon)
Department of Mathematical Sciences
Seoul National University, Seoul 08826, Republic of Korea
Email address: jyoung924@snu.ac.kr