Quantum Teleportation using Quantum Information Theory

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Definition (Classical Entropy, Shannon Entropy)

When the probability distribution of a random variable X is given as $p(x)$, its Shannon Entropy is defined as

$$S(X) = -\sum_x p(x) \log p(x).$$

Remark. The Shannon Entropy of a random variable represents the uncertainty. The entropy is always non-negative, and is zero when $p(x) = 1$. Among the distributions of fixed domain, the entropy obtains the maximal value at the uniform probability distribution.

Remark. Viewing qubits as some random variable and applying the Shannon Entropy is problematic. This is because the probability distribution differs from the choice of the measurement basis. In other words, defining the entropy using some probability distribution of measurement is not well defined. Thus, there's a need for novel definition of entropy for qubits, that are irrelevant from the choice of the measurement basis.

Idea. Since the eigen-decomposition of the matrix is unique, when

$$\rho = \sum_j p_j |j\rangle\langle j|,$$

define the entropy as the Shannon entropy of $S(p_j, j = 1, 2, \cdots)$.

Basics of Quantum Information Theory

Definition (Quantum Entropy, von Neumann Entropy)

When the density matrix of a system A is given as ρ_A, its Quantum Entropy is defined as

$$S_A = S(\rho_A) = -\operatorname{tr}\left[\rho_A \log \rho_A\right].$$

Definition (Quantum conditional entropy)

When the system AB is given, the quantum conditional entropy is computed as

$$S(A|B) = S_{AB} - S_B.$$

Basics of Quantum Information Theory

Here, we review some key properties about quantum entropy

Theorem

- *Non-negativity :* $S(A) \geq 0$.
- *Pure state entropy : A is a pure state if and if only* $S(A) = 0$.
- *Concavity :* $\sum_i p_i S(\rho_i) \leq S\left(\sum_i p_i \rho_i\right)$

Remark. While the conditional entropy is also non-negative in classical entropy, it is not in the quantum. For example, when AB is a pure state,

$$S(A|B) = S_{AB} - S_B = 0 - S_B < 0,$$

since B is in a mixed state.

Basics of Quantum Information Theory

Here is an important property of the quantum entropy about pure state that does not have a classical analog.

Theorem

When a system AB forms a pure state, then the following equality holds.

$$S_A = S_B.$$

Proof.

The state AB can be written as the following since its pure :

$$\psi_{AB} = \sum_i \sqrt{p_i} \psi_A^i \otimes \psi_B^i.$$

Compute the density matrix of A and B,

$$\rho_A = \sum_i \sqrt{p_i} \psi_A^i, \qquad \rho_B = \sum_i \sqrt{p_i} \psi_B^i.$$

Thus, $S_A = S_B$. □

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Quantum Teleportation

Goal. Send an arbitrary Qubit A_0 from Alice to Bob.

Setting. Alice and Bob share an entangled pair A_1B.

Solution. When $A_0 = \alpha\left|0\right\rangle + \beta\left|1\right\rangle$, the initial state is written as

$$\Psi_{A_0A_1B} = \frac{1}{\sqrt{2}}\left(\alpha\left|000\right\rangle + \alpha\left|011\right\rangle + \beta\left|100\right\rangle + \beta\left|111\right\rangle\right).$$

Then, Alice measures her qubits A_0A_1 to the following basis :

$$\{\left|\phi_i\right\rangle | i = 0, 1, 2, 3\} = \left\{\frac{1}{\sqrt{2}}\left(\left|00\right\rangle \pm \left|11\right\rangle\right), \frac{1}{\sqrt{2}}\left(\left|01\right\rangle \pm \left|10\right\rangle\right)\right\}$$

Finally, Bob applies X or Z gates on B to obtain the initial A_0, according to results of Alice's measurement.

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State Merging

Setting.

- A system R has a purification RAB.
- Alice holds the state A.
- Bob holds the state B.

Goal.

1. Alice measures A on some bases, namely $\{E_i\}$.
2. Alice send the measured result to Bob.
3. Bob generates a system A' with $\dim A = \dim A'$.
4. Bob applies an Unitary gate U_a on $A'B$.

Within these steps, the goal is to make $\Psi^i_{RAB} = \Psi^f_{RA'B}$.

Question. Would it be possible to make Bob to obtain all of the purification? i.e., Such E_i and U_a exist?

State Merging

Notation. For the simplicity, the superscript i, a, f represents the initial, after the measurement of Alice, final state, respectively. i.e.,

1. Ψ^i_{RAB} is an initial pure state.
2. Alice measures A and obtains $|i\rangle_A$, and the state is $\Psi^a_{RB} \bigotimes |i\rangle_A$.
3. Bob generates the system A' as $|i\rangle_{A'}$, hence the state is $\Psi^a_{RB} \bigotimes |i\rangle_{A'}$.
4. Bob applies an Unitary gate U_a on $A'B$ and obtain $\Psi^f_{RA'B}$.

Remark. States $\Psi^a_{RB} \bigotimes |i\rangle_{A'}$ and $\Psi^f_{RA'B}$ are also both purification of R. Thus, existence of U_a is equivalent to ρ_R staying constant.

State Merging

Theorem

If the state merging is possible, then $S\left(A|B\right) \leq 0$.

Proof.

Since ρ_R stays constant, so does S_R. For the remark, $S_R = S_{AB}$ since RAB forms a pure state, and so does $S(\rho_B^a)$ after the measurement. Thus, $S_{AB} = S_{\rho_B^a}$ holds, regardless of the outcome of the measurement. Since ρ_B before and after the measurement is :

$$\rho_B^i = \mathrm{tr}_{RA}\,\rho_{RAB}^i$$
$$\rho_B^a = \frac{1}{p_a}\mathrm{tr}_{RA}\,\pi_A^a\rho_{RAB}^i\pi_A^a = \frac{1}{p_a}\mathrm{tr}_{RA}\,\pi_A^a\rho_{RAB}^i,$$

ρ_B^i can be expressed as

$$\rho_B^i = \Sigma_a p_a \rho_B^a.$$

State Merging

Proof (Cont.)

By the concavity of the entropy,

$$S(\rho_B^i) \geq \Sigma_a p_a S(\rho_B^a).$$

Express S_{AB} as

$$S_{AB} = \Sigma_a p_a S(\rho_{AB}) = \Sigma_a p_a S(\rho_B^a).$$

Thus, we finish the proof by

$$\begin{aligned} S\left(A|B\right) &= S_{AB} - S(B) \\ &\leq \Sigma_a p_a S(\rho_B^a) - \Sigma_a p_a S(\rho_B^a) = 0. \end{aligned}$$

□

State Merging

Corollary

When $S(A|B) > 0$, at least extra $S(A|B)$ number of maximally entangled pairs should be shared to make state merging possible.

Proof.

Assume that n maximally entangled pairs, namely $\tilde{A}\tilde{B}$, are shared between Alice and Bob. With R's purification being $RA\tilde{A}B\tilde{B}$,

$$\begin{aligned} S\left(A\tilde{A}\middle|B\tilde{B}\right) &= S\left(A\tilde{A}B\tilde{B}\right) - S\left(B\tilde{B}\right) \\ &= S_{AB} + S_{\tilde{A}\tilde{B}} - S_B - S_{\tilde{B}} \\ &= S_{AB} + 0 - S_B - n \\ &= S\left(A|B\right) - n. \end{aligned}$$

Since $S\left(A\tilde{A}\middle|B\tilde{B}\right) \leq 0$ to allow state merging, $S\left(A|B\right) \leq n$. □

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Non-maximally entangled pair

Question. The conditional entropy of the shared entangled qubits $S(A_1|B)$ is related to the availability of quantum teleportation when A_1, B are maximally entangled state. However, being maximally entangled is an ideal situation. **How are the conditional entropy $S(A_1|B)$ related to the quantum transportation if $A_1 B$ are in mixed state?**

Setting. We will use the setting where $A_1 B$ are mixed with the vacuum noise.

$$\rho_{A_1 B} = p \left|\psi^+\middle\rangle\middle\langle\psi^+\right| + \frac{1}{4}(1-p)I$$

$$\rho_B = \operatorname{tr}_{A_1} \rho_{A_1 B} = \frac{1}{2} I$$

Non-maximally entangled pair

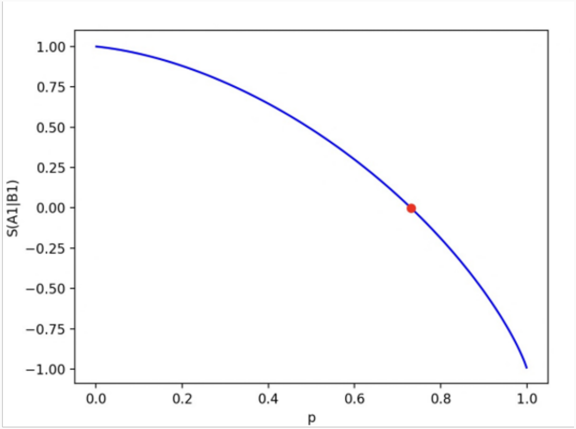
Remark. The quantum entropy of ρ_{A_1B} and B are

$$S(\rho_{A_1B}) = -\frac{1+3p}{4}\log\frac{1+3p}{4} - \frac{3(1-p)}{4}\log\frac{1-p}{4}$$
$$S(\rho_B) = 1.$$

The computation is done by decomposition through POVM set.
Remark. The conditional entropy $S(A_1|B)$ is :

$$\begin{aligned} S(A_1|B) &= S(A_1B) - S(B) \\ &= -\frac{1+3p}{4}\log\frac{1+3p}{4} - \frac{3(1-p)}{4}\log\frac{1-p}{4} - 1. \end{aligned}$$

Non-maximally entangled pair



$p > 0.748$

$\rightarrow S(A_1|B) < 0$

$p < 0.748$

$\rightarrow S(A_1|B) > 0$

Non-maximally entangled pair

Remark. It is impossible to perform quantum teleportation with fidelity=1 if the mixed state is used, even when $S(A_1|B) < 0$. This is because purification R of $A_1 B$ collapses to certain state through the measurement of $A_0 A_1$, changing $S(R)$.

Proof.

Without the loss of generality, assume A_0 is set as $|0\rangle$. Then, the density matrix of AB is

$$\rho_{AB} = |0\rangle\langle 0|_{A_0} \otimes \left(p \left|\psi^+\right\rangle\left\langle\psi^+\right|_{A_1 B} + \frac{1}{4}(1-p) I_{A_1 B} \right).$$

The purification R needs orthogonal basis of 4, referring as

$$|0\rangle_R \,, |1\rangle_R \,, |2\rangle_R \,, |3\rangle_R \,.$$

Non-maximally entangled pair

Proof (Cont.)

Then the state RAB can be expressed as :

$$|\psi\rangle_{RAB} = |0\rangle_{A_0} \left(\frac{\sqrt{1+3p}}{2} \left|\psi^+\right\rangle_{A_1B} |0\rangle_R + \frac{\sqrt{1-p}}{2} \left|\psi^-\right\rangle_{A_1B} |1\rangle_R + \frac{\sqrt{1-p}}{2} \left|\phi^+\right\rangle_{A_1B} |2\rangle_R + \frac{\sqrt{1-p}}{2} \left|\phi^-\right\rangle_{A_1B} |3\rangle_R \right).$$

Thus the entropy of R is equivalent to the Shannon Entropy

$$S_R = S\left(\frac{1-3p}{4}, \frac{1+p}{4}, \frac{1+p}{4}, \frac{1+p}{4}\right).$$

Non-maximally entangled pair

Proof (Cont.)

On the other hand, rewrite the state RAB by expressing A into linear combination of the Bell states :

$$|\psi\rangle_{RAB} = \left|\psi^+\right\rangle_A \left(\frac{\sqrt{1+3p}}{2}\,|00\rangle + \frac{\sqrt{1-p}}{2}\,(|01\rangle + |12\rangle + |13\rangle)\right)_{BR} + \left|\psi^-\right\rangle_A (\cdots) + \left|\phi^+\right\rangle_A (\cdots) + \left|\phi^-\right\rangle_A (\cdots)$$

Then, perform Bell measurement on A and for example, obtain $|\psi^+\rangle$. In this case, after the measurement,

$$|\psi^a\rangle_{BR} = \left(\frac{\sqrt{1+3p}}{2}\,|00\rangle + \frac{\sqrt{1-p}}{2}\,(|01\rangle + |12\rangle + |13\rangle)\right)_{BR}.$$

Non-maximally entangled pair

Proof (Cont.)

The density matrix of R after the measurement is

$$\begin{aligned}\rho_R^a &= \operatorname{tr}_{AB} |\psi^a\rangle\langle\psi^a|_{RAB} \\ &= \begin{bmatrix} \begin{bmatrix} \frac{1+3p}{4} & \frac{\sqrt{(1+3p)(1-p)}}{4} \\ \frac{\sqrt{(1+3p)(1-p)}}{4} & \frac{1-p}{4} \end{bmatrix} & O \\ O & \begin{bmatrix} \frac{1-p}{4} & \frac{1-p}{4} \\ \frac{1-p}{4} & \frac{1-p}{4} \end{bmatrix} \end{bmatrix},\end{aligned}$$

making the entropy of R after the measurement as

$$S(\rho_R^a) = S\left(\frac{1+p}{2}, \frac{1-p}{2}, 0, 0\right) \neq S(R).$$

□

Non-maximally entangled pair

Remark. However, the conditional entropy is related to the Fidelity.

Definition (Fidelity)

Given two density matrices ρ_ϕ and τ, the fidelity is defined as a quantity

$$f = \mathrm{tr}\left[\left(\sqrt{\tau}\rho_\phi\sqrt{\tau}\right)^{\frac{1}{2}}\right].$$

The average fidelity is defined as

$$F = \int d\phi f$$

Non-maximally entangled pair

Recall. When $A_1 B$ are entangled with the vacuum noise,

$$\rho_{A_1 B} = p \, |\psi\rangle\langle\psi| + \frac{1}{4}(1-p) I_4.$$

The density matrix of the system AB as a whole is $\rho_{AB} = |\phi\rangle\langle\phi| \otimes \rho_{A_1 B}$.

Then, after the measurement, the density matrix of AB can be written as

$$\rho^a_{AB} = \sum_{\alpha=0}^{3} (I_4 \otimes U_\alpha) \, \langle\psi_\alpha|\rho_{AB}|\psi_\alpha\rangle \left(I_4 \otimes U^\dagger_\alpha\right).$$

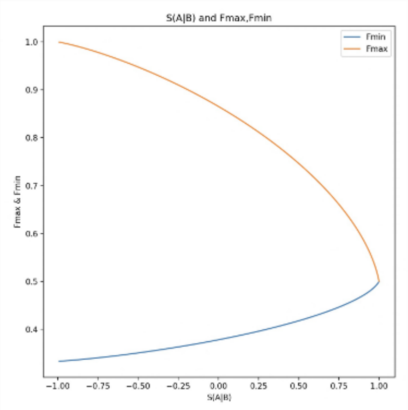
As a result, the fidelity between ρ_{AB} and ρ^a_{AB} is

$$f = \frac{p}{4} \sum_{\alpha=0}^{3} \xi_\alpha + \frac{1}{2}(1-p).$$

Non-maximally entangled pair

Result. Thus, we have

$$\left[F_{min} = \frac{1}{2}\left(1 - \frac{p}{3}\right)\right] \leq F \leq \left[F_{max} = \frac{1}{2}(1+p)\right]$$



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Conclusion

- Quantum conditional entropy of the shared entangled pair is closely related to the Quantum teleportation.
- When the shared state A_1B is a pure state, the conditional entropy is related to the availability of the quantum teleportation.
- When the shared state A_1B is in a mixed state, the conditional entropy is related to the fidelity when the quantum teleportation is performed.

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