Seminar : Automata Theory

Lec 1 : Equivalency of Finite Automata and Regular expression

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Language

Definition (Alphabet)

Alphabet is a finite set of characters. We denote Σ as a alphabet set.

Example. Following sets are examples of the alphabet.

- $\{0, 1\}$
- $\{a, b, c, \cdots, z\}$

Definition (String)

String generated by Σ is a finite length word that uses characters in Σ. We denote the length of the string w as $|w|$. We say a string is an empty string if it is of length zero. We denote the empty string as ϵ.

Language

Definition (Concatenation)

When the two strings x, y are given, we denote a concatenation of as xy. When the two sets of strings X, Y are given, we define a concatenation of sets X, Y as

$$XY = \{xy : x \in X, y \in Y\}.$$

Definition (Kleene product)

For a set of strings S, we denote a set S^* as a Kleene product of the set S :

$$S^* = \bigcup_{k=0}^{\infty} S^k, \quad S^{k+1} = S^k S, \quad S^0 = \{\epsilon\}$$

Remark. Note that the set of all strings generated by the set Σ is Σ^*. For example, when $\Sigma = \{0, 1\}$, then

$$\Sigma^* = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, 001, \cdots\}.$$

Language

Definition (Language)

- We say a set L is a *Language* if $L \subset \Sigma^*$.
- We call a set of Languages as *Language Class*.

Remark. Since Language is a set of strings, we can define concatenation and Kleene product identically on Languages.

Remark. Definition of *Language* coincides with the general notion of languages such as English, Korean, C, Python.

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Regular Expression RE

Goal. In this section we define the first useful Language class, called *RE*.

Definition (Regular Expression)

We denote a language class *Regular Expression* as RE. Language class RE is defined inductively using following rules.

1. $\emptyset \in RE$: An empty set $\emptyset$ is regular expression.
2. $\epsilon \in RE$: A singleton set $\epsilon = \{\epsilon\}$ is regular expression.
3. $a \in \Sigma \Rightarrow a \in RE$: A singleton set $a = \{a\}$ is regular expression for every alphabet $a \in \Sigma$.
4. $r, s \in RE \Rightarrow (r + s), (rs), (r^*) \in RE$: When r, s are regular expression and equivalent to the set of strings R, S, $(r + s), (rs), (r^*)$ respectively denote sets of strings $R \cup S, RS, R^*$ and are all regular expressions.

Remark. We denote a set of strings expressed by $r \in RE$ as $L(r)$.

Regular Expression RE

Exmaples. Here's some examples of the class RE.

- $((0+1)(0+1)(0+1))^*$: Set of strings with length is a multiple of 3.
- $(1+\epsilon)(0^*01)^*0^*$: Set of string without 11 in the string.
- $(00)^*(11)^* + 0(00)^*1(11)^*$: Set of string of form 0^n1^m, $2|n+m$.

Definition (Equivalent)

We say some form of the expression A, B is equivalent if its corresponding sets satisfy $L(A) = L(B)$.

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Deterministic Finite Automata DFA

Goal. In this section we define another Language class, called *DFA*.

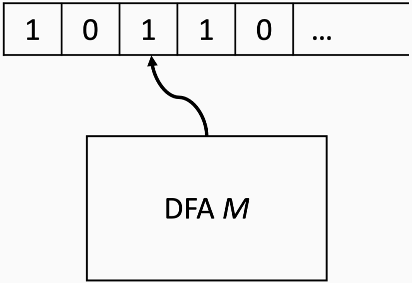
Definition (Deterministic Finite Automata)

We denote a language class *Deterministic Finite Automata* as *DFA*.
Language $M = (Q, \Sigma, \delta, q_0, F)$ is said to be in *DFA* if it is composed with

1. Q : Finite set of *States*.
2. Σ : The *Alphabet* set.
3. $\delta : Q \times \Sigma \to Q$: *Transfer function* that chooses the next state.
4. $q_0 \in Q$: The *Initial state*.
5. $F \subseteq Q$: The *Final states*.

Deterministic Finite Automata DFA

Remark. Deterministic Finite Automata runs by two extra devices, called *Head* and *Tape*. *Tape* contains the input string, while *Head* moves far left to far right reading the input string.



When the current state is $q \in Q$, and the *Head* reads the alphabet $a \in \Sigma$, the next state is $\delta(q, a)$ and the head moves right and reads the next character in the *Tape*.

Deterministic Finite Automata DFA

Notation. We describe state transfer using $\vdash_M$. When our current configuration (i.e. current state and unread part of string) is $(q, 110)$ and if $\delta(q, 1) = q'$ then

$$(q, 110) \vdash_M (q', 10).$$

If clear, omit M and use $\vdash$. 0 or more steps of transfer is denoted as $\vdash^*$.

Definition

We say a string $\omega \in \Sigma^*$ is *accepted* by DFA M if DFA M with input ω ends in the *Final state* F. We define the language $L(M)$ as the set of strings that are accepted by M :

$$L(M) = \{\omega \in \Sigma^* : (q_0, w) \vdash^* (f, \epsilon), f \in F\}.$$

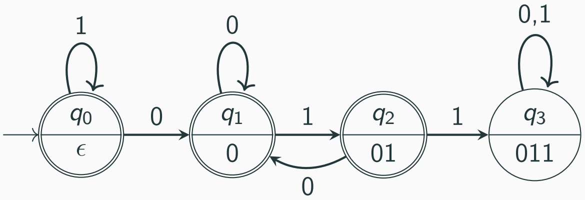
Define a map $\delta^* : Q \times \Sigma^* \to Q$ inductively using $\delta^*(q, \epsilon) = q$ and $\delta^*(q, \omega a) = \delta(\delta^*(q, \omega), a)$ for all $\omega \in \Sigma^*, a \in \Sigma, q \in Q$. Then,

$$L(M) = \{\omega \in \Sigma^* : \delta^*(q_0, \omega) \in F\}.$$

Deterministic Finite Automata DFA

Exmaples. Here's some examples of the class *DFA*.

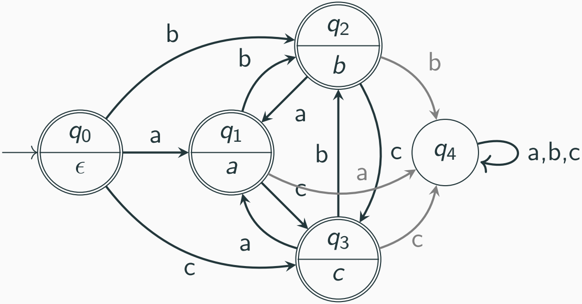
- Set of string without 011 in the string.



Deterministic Finite Automata DFA

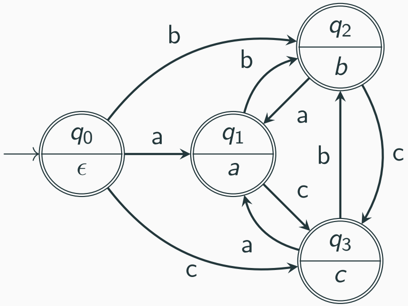
Exmaples. Here's some examples of the class *DFA*.

- Set of string without identical consecutive characters. $\Sigma = \{a, b, c\}$.



Remark. For the simplicity, we allow δ to be a *partial function*. In such case, we append a new state $q' \notin F$ so we can extend δ as

$$\delta(q, a) = \begin{cases} \delta(q, a) & \text{if } \delta(q, a) \text{ is defined} \\ q' & \text{if not defined or } q = q' \end{cases}$$



We call q', which was q_4 in the previous slide as *dead state*.

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Nondeterministic Finite Automata NFA

Goal. In this section we define another Language class, called *NFA*.

Definition (Nondeterministic Finite Automata)

We denote a language class *Nondeterministic Finite Automata* as *NFA*. Language $M = (Q, \Sigma, \Delta, q_0, F)$ is said to be in *NFA* if it is composed with

1. Q : Finite set of *States*.
2. Σ : The *Alphabet* set.
3. $\Delta : Q \times (\Sigma \cup \epsilon) \rightrightarrows Q$: *Transfer relation*, collection of next state(s).
4. $q_0 \in Q$: The *Initial state*.
5. $F \subseteq Q$: The *Final states*.

Nondeterministic Finite Automata NFA

Remark. Nondeterministic Finite Automata is a generalization of *DFA* :

- it can have multiple next states or no next state.
- it can transfer with the empty string ϵ.

Remark. We may view Δ as a function of $Q \times (\Sigma \cup \epsilon) \to 2^Q$. In this sense, extend Δ as $\Delta(P, a) = \cup_{q \in P} \Delta(q, a)$ for $P \in 2^Q$.

Remark. We now define a set of states $E(q)$ for each state q. $E(q)$ stands for the states reachable from q without reading any string :

$$E(q) = \bigcup_{i \in \mathbb{Z}_{\geq 0}} E^i(q), \quad E^0(q) = q, \quad E^{k+1}(q) = \Delta(E^k(q), \epsilon).$$

Remark. While we can define $\vdash$ similarly, we should note that

transferred result need not be unique.

Nondeterministic Finite Automata NFA

Definition

We say a string $\omega \in \Sigma^*$ is *accepted* by NFA M if NFA M with input ω ends in the *Final state* F. We define the language $L(M)$ as the set of strings that are accepted by M :

$$L(M) = \{\omega \in \Sigma^* : (q_0, w) \vdash^* (f, \epsilon), f \in F\}.$$

Define a map $\Delta^* : Q \times \Sigma^* \to 2^Q$ inductively using $\Delta^*(q, \epsilon) = E(q)$ and $\Delta^*(q, \omega a) = E(\Delta(\Delta^*(q, \omega), a))$ for all $\omega \in \Sigma^*, a \in \Sigma, q \in Q$. Then,

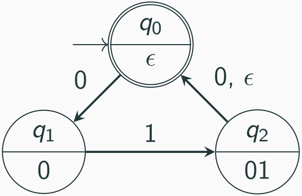
$$L(M) = \{\omega \in \Sigma^* : \Delta^*(q_0, \omega) \cap F \neq \emptyset\}.$$

Remark. We may assume that $|F| = 1$. If $|F| > 1$, then append new state f, use $\{f\}$ as a *Final state*, and append $\Delta(q, \epsilon) = f$ for all $q \in F$.

Nondeterministic Finite Automata NFA

Exmaples. Here's some examples of the class *NFA*.

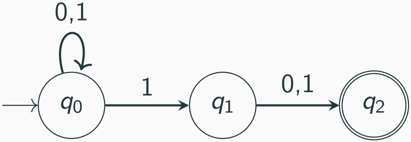
- Regular expression $(010 + 01)^*$.



Nondeterministic Finite Automata NFA

Exmaples. Here's some examples of the class *DFA*.

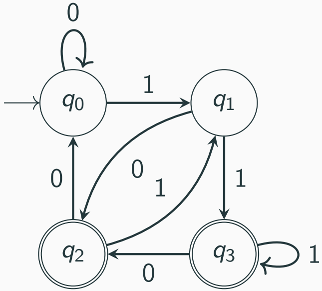
- Set of string with 1 as the second last character.



Remark. It is trivial that *DFA* is an instance of *NFA*. Thus, if a language L is of class *DFA*, then it is also *NFA*.

Remark. Generally, it is much simpler to build *NFA* than *DFA*.

- Set of string with 1 as the second last character (*DFA*).



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$DFA = NFA$

Goal. In this section we prove $DFA = NFA$.

Remark. We already know $DFA \subseteq NFA$.

Idea. The idea of proving $NFA \subseteq DFA$ is to build an equivalent DFA for a given NFA with each subset of 2^Q as a state of DFA.

Theorem

$$DFA = NFA$$

To be precise, for a language L, there exists a DFA that accepts strings in L and denies others, if and if only when there exists a NFA that accepts strings in L and denies others.

DFA $=$ *NFA*

proof of $NFA \subseteq DFA$**.**

Suppose *NFA* $M(Q, \Sigma, \Delta, q_0, F)$ is given. We aim to build an equivalent *DFA*. Consider *DFA* machine $M_D(Q_D, \Sigma, \delta, q_D, F_D)$ as :

1. $Q_D = 2^Q$.
2. $\delta : 2^Q \times \Sigma \to 2^Q$, such that $\delta(P, a) = E(\Delta(P, a))$ for $P \subseteq Q$.
3. $q_D = E(q_0)$.
4. $F_D = \{P : P \cap F \neq \emptyset\}$.

Then we can easily conclude the proof by showing that a string is accepted by M if and only when it is accepted by M_D. The proof is done via induction on the length of the input string. $\square$

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Goal. In this section we prove that RE is equivalent to finite automata.

Idea. We first prove that $RE \subseteq NFA$ and then prove $DFA \subseteq RE$.

Theorem

$$RE = DFA = NFA$$

To be precise, for a language L, it is describable as Regular expression if and only if when there exists a DFA (of NFA) that accepts strings in L and denies others.

Idea. For $RE \subseteq NFA$, we will build NFA.

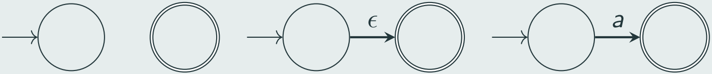
Idea. For $DFA \subseteq RE$, we will construct RE.

$RE = (DFA = NFA)$

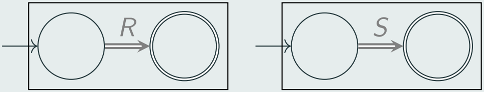
proof of $RE \subseteq NFA$.

Recall that RE is defined inductively. We construct NFA with *singleton Final state* corresponding to RE also inductively.

1. First, $\emptyset, \epsilon, a \in RE$ for $a \in \Sigma$. Corresponding NFA are :

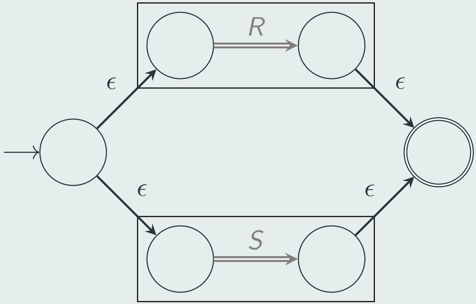


2. $r, s \in RE \Rightarrow (r + s), (rs), (r^*) \in RE$:
   Assume r, s have corresponding NFA : R, S :



proof of $RE \subseteq NFA$, continued.

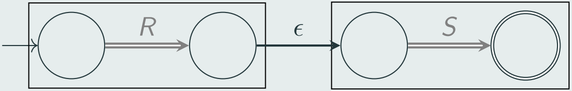
Then NFA for $(r + s)$ is :



NFA for $(r + s)$

proof of $RE \subseteq NFA$, continued.

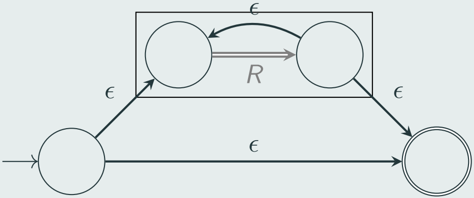
NFA for (rs) is :



NFA for (rs)

proof of $RE \subseteq NFA$, continued.

NFA for (r^*) is :



NFA for (r^*)

$\square$

Now let's prove $DFA \subseteq RE$. The idea is to categorize the strings via the states it passes.

proof of $DFA \subseteq RE$.

Let's write the given DFA M as :

$$M = (Q, \Sigma, \delta, q_1, F), \quad Q = \{q_1, q_2, \cdots, q_n\}.$$

Let's define a set of strings R_{ij}^k as

$$R_{ij}^k = \left\{ \omega \in \Sigma^* : \text{when } (q_i, \omega) \vdash^* (q_m, \omega'), \begin{cases} m \le k & \text{if } |\omega'| > 0 \\ m = j & \text{if } \omega' = \epsilon \end{cases} \right\},$$

strings which transfers q_i to q_j without passing any states with index number larger than k.

proof of $DFA \subseteq RE$, continued.

For the remark,

$$L(M) = \bigcup_{q_j \in F} R_{1j}^n.$$

The set R_{ij}^k could be calculated recursively as follows :

$$R_{ij}^0 = \begin{cases} \{a \in \Sigma : \delta(q_i, a) = q_j\} & \text{if } i \neq j \\ \{a \in \Sigma : \delta(q_i, a) = q_i\} \cup \epsilon & \text{if } i = j \end{cases}$$

$$R_{ij}^k = R_{ij}^{k-1} \cup R_{ik}^{k-1}(R_{kk}^{k-1}) * R_{kj}^{k-1}.$$

proof of $DFA \subseteq RE$, continued.

In other words, each R_{ij}^k are a RE language r_{ij}^k defined recursively as :

$$r_{ij}^0 = \begin{cases} \{a \in \Sigma : \delta(q_i, a) = q_j\} & \text{if } i \neq j \\ \{a \in \Sigma : \delta(q_i, a) = q_i\} + \epsilon & \text{if } i = j \end{cases}$$

$$r_{ij}^k = r_{ij}^{k-1} + (r_{ik}^{k-1}(r_{kk}^{k-1}) * r_{kj}^{k-1}).$$

Thus, $L(M)$ is RE language since

$$L(M) = \sum_{q_j \in F} r_{1j}^n.$$

$\square$

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Theorem (Pump theorem)

Suppose L is a RE language. Then, there exist an integer $t > 0$ such that, for any $\omega \in L$ with $|\omega| \geq t$, it is possible to decompose ω as xyz that satisfies :

1. $|xy| \leq t$ *and* $|y| \geq 1$,
2. *For all* $i \geq 0$, $xy^i z \in L$.

Idea. Find equivalent *DFA* and define t as the number of states.
Remark. The inverse does not hold.

proof of Pump theorem.

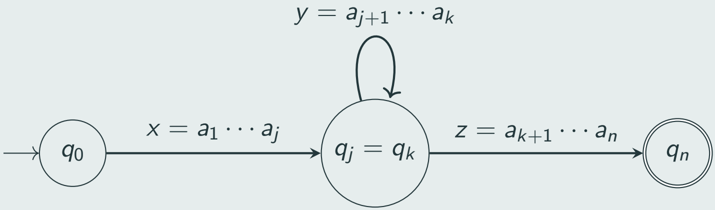
Now we can consider equivalent *DFA* $M = (Q, \Sigma, \delta, q_0, F)$. Choose the number t as $|Q|$, which only depends on the language L. Name the characters of the input string as $\omega = a_1 a_2 \cdots c_n$ where $|\omega| = n \geq t$. Label the state M stays as it reads the input as

$$q_i = \delta^*(q_0, a_1 a_2 \cdots a_i), \quad 1 \leq i \leq n.$$

With the pigeon's hole principle, among $q_0, q_1, \cdots, q_t$ there exists $0 \leq j < k \leq t$ such that $q_j = q_k$.

Further topics

proof of Pump theorem, continued.



Define x, y, z as the figure above. Then, $|xy| = k \leq t$ and $|y| = k - j \geq 1$. Also, from

$$q_j = \delta^*(q_j, y),$$

we can say $q_j = \delta^*(q_j, y^*)$. Hence, $\delta^*(q_0, xy^i z) = q_n \in F$.
$\therefore xy^i z \in L, \quad \forall i \geq 0.$ $\square$

Examples of languages not in *RE*

- $\{0^n 1^n : n \geq 0\}$.
- $\{uu : u \in \Sigma^*\}$. $(|\Sigma| \geq 2)$
- $\left\{a^{n^2} : n \geq 0\right\}$.
- $\{0^m 1^n : m \neq n\}$.

Remark. What is *RE* or *FA* in practical issue? We can see that *RE* are not capable of recording a long term memory, while short term memory of bounded length is possible. In the example, It was not possible to duplicate a sequence after it is once written, while it is possible to check the character written right before.

Remark. Due to such characteristics, *FA*s are commonly refered as the computer with out a memory device. We use them even in real life, such as a door lock.